Multidimensional Fractional Programming for Normalized Cuts

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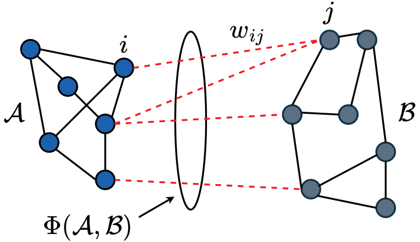
Normalized Cut Problem



- Similarity between i and j is quantified by the weight $w_{ij} \in [0, 1]$.
- The cut between two clusters is

$$\Phi(\mathcal{A}, \mathcal{B}) = \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{B}} w_{ij}$$

- We consider minimizing a normalized cut across K clusters:

$$\mathrm{ncut}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_K) = \sum_{k=1}^{K} \frac{\Phi(\mathcal{V}_k, \bar{\mathcal{V}}_k)}{\mathrm{vol}(\mathcal{V}_k)}.$$

Normalized Cut Problem

After some algebra, the NCut problem can be rewritten as

$$\begin{aligned} \underset{\{x_{ik}\}}{\text{maximize}} \quad & \sum_{k=1}^{K} \frac{\boldsymbol{x}_k^\top \boldsymbol{W} \boldsymbol{x}_k}{\boldsymbol{x}_k^\top \boldsymbol{D} \boldsymbol{x}_k} \\ \text{subject to} \quad & \sum_{k=1}^{K} x_{ik} = 1, \quad i = 1, \ldots, n \\ & x_{ik} \in \{0, 1\}, \quad i = 1 \ldots, n, \ k = 1, \ldots . K \end{aligned}$$

This problem is fractionally structured, we propose using fractional programming techniques to solve it.

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Classical FP Technique

Proposition (Dinkelbach's transform)

The single-ratio problem

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad \frac{A(x)}{B(x)}$$

is equivalent to

$$\underset{x \in \mathcal{X}}{\text{maximize}} \quad A(x) - yB(x),$$

where the auxiliary variable y is iteratively updated as $y = A(x)/B(x)$.

Dinkelbach's transform only works for single-ratio problems, and does not work for the NCut with general K clusters.

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SOTA FP Technique

Proposition (Quadratic transform)

The multi-ratio problem

$$\underset{x \in \mathcal{X}, \boldsymbol{y}}{\textit{maximize}} \quad \sum_{k=1}^{K} \frac{A_k(x)}{B_k(x)}$$

is equivalent to

$$\underset{x \in \mathcal{X}, y}{\textit{maximize}} \quad \sum_{k=1}^{K} 2y_k \sqrt{A_k(x)} - y_k^2 B_k(x),$$

where the auxiliary variables $\{y_k\}$ *are iteratively updated as* $y_k = \sqrt{A_k(x)}/B_k(x)$.

[3] K. Shen and W. Yu, "Fractional programming for communication systems—Part I: Power control and beamforming," *IEEE Trans. Signal Process.*, vol. 66, no. 10, pp. 2616–2630, Mar. 2018.

SOTA FP Technique

- By quadratic transform, we can recast the NCut problem into

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\underset{\{x_{ik}, y_k\}}{\text{maximize}} \quad & \sum_{k=1}^{K} 2y_k\sqrt{\boldsymbol{x}_k^\top \boldsymbol{W}\boldsymbol{x}_k} - y_k^2\boldsymbol{x}_k^\top \boldsymbol{D}\boldsymbol{x}_k \\
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- *The numerators and the denominators are decoupled. But the problem is still difficult to solve.*

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Matrix FP Problem

- Each $A_k(x) \geq 0 \implies \boldsymbol{A}_k(x) \in \mathbb{S}_+^{m \times m}$.
- Each $B_k(x) > 0 \implies \boldsymbol{B}_k(x) \in \mathbb{S}_{++}^{m \times m}$.
- The ratio term is then extended to the matrix form as

$$\frac{A_k(x)}{B_k(x)} \in \mathbb{R}_+ \implies \boldsymbol{B}_k(x)^{-1} \boldsymbol{A}_k(x) \in \mathbb{S}_+^{m \times m}.$$

- The matrix version of the sum-of-ratios problem is

$$\underset{x \in \mathcal{X}}{\text{maximzie}} \quad \sum_{k=1}^{K} \mathrm{tr}\left(\boldsymbol{B}_k^{-1}(x) \boldsymbol{A}_k(x)\right).$$

New FP Technique

Proposition (Multidimensional quadratic transform)

Suppose that each $\boldsymbol{A}_k(x) \in \mathbb{S}_+^{m\times m}$ *can be factorized as*

$$\boldsymbol{A}_k(x) = [\boldsymbol{Z}_k(x)]^\top [\boldsymbol{Z}_k(x)] \quad \textit{where} \;\; \boldsymbol{Z}_k(x) \in \mathbb{R}^{\ell\times m}$$

for some positive integer ℓ.

The matrix FP problem is equivalent to

$$\underset{x\in\mathcal{X},\,\boldsymbol{Y}_k\in\mathbb{R}^{\ell\times m}}{\textit{maximize}} \quad \sum_{k=1}^{K} \operatorname{tr}\left(2\boldsymbol{Y}_k[\boldsymbol{Z}_k(x)]^\top - \boldsymbol{Y}_k\boldsymbol{B}_k(x)\boldsymbol{Y}_k^\top\right).$$

New Problem Formulation

- By the multidimensional quadratic transform, we reformulate the NCut problem as

$$\begin{aligned} \underset{\{x_{ik},\boldsymbol{y}_k\}}{\text{maximize}} \quad & \sum_{k=1}^{K}\left(2\boldsymbol{y}_k^\top \boldsymbol{W}^{\frac{1}{2}} - \boldsymbol{y}_k^\top \boldsymbol{y}_k \boldsymbol{\delta}^\top\right)\boldsymbol{x}_k \\ \text{subject to} \quad & \sum_{k=1}^{K} x_{ik} = 1, \quad i = 1,\ldots,n \\ & x_{ik} \in \{0,1\}, \quad i = 1\ldots,n,\ k = 1,\ldots.K \end{aligned}$$

- When $\{x_{ik}\}$ are fixed, the auxiliary variables $\{\boldsymbol{y}_k\}$ can be optimally updated as $\boldsymbol{y}_k = \frac{\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{x}_k}{\boldsymbol{x}_k^\top \boldsymbol{D}\boldsymbol{x}_k}$.
- When $\{\boldsymbol{y}_k\}$ are fixed, the new problem is a bipartite problem of $\{x_{ik}\}$, which can be efficiently solved by the standard method.

Numerical Results

Table: NCut objective values achieved by the different algorithms. The best results are marked in bold.

	SC	FINC	FCD	FPC
Breast	2.438568	2.445278	2.446499	**2.431813**
Thyroid	0.983144	0.983393	0.986163	**0.983115**
Office+Caltech10	4.483921	4.483962	4.491925	**4.483501**
Splice	0.997651	0.999867	0.998569	**0.997636**
Rice	**0.499193**	0.499999	0.499209	**0.499193**
Landsat	2.994675	2.999986	2.995302	**2.994335**
USPS	4.476479	4.476404	4.476591	**4.475869**
Epileptic	1.992376	1.991369	1.992688	**1.991311**

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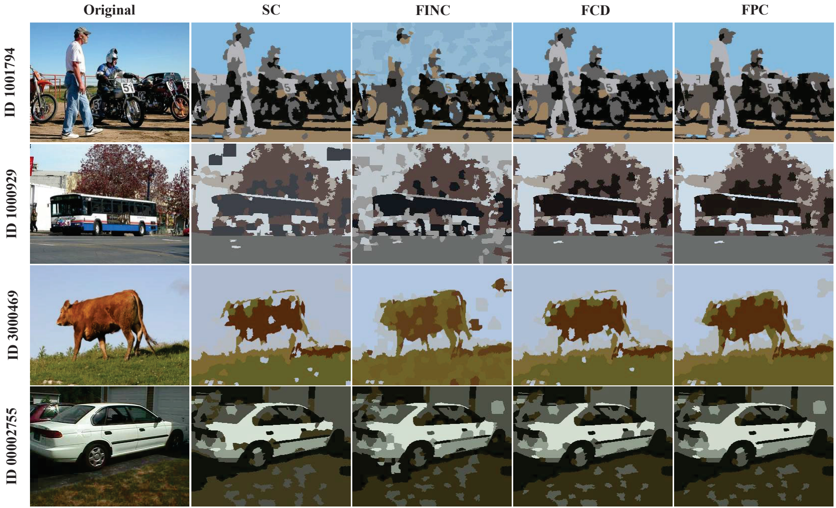

Figure: Image segmentation by the different algorithms.

Recap

- The NCut problem is fractionally structured, recognized as a matrix-ratio fractional program.
- Dinkelbach's algorithm is limited to single ratio, so it does not work for NCut.
- We propose using a new technique called matrix quadratic transform to recast NCut to a sequence of bipartite matching problems.

Thanks!

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