Learning Cut Generating Functions for Integer Programming

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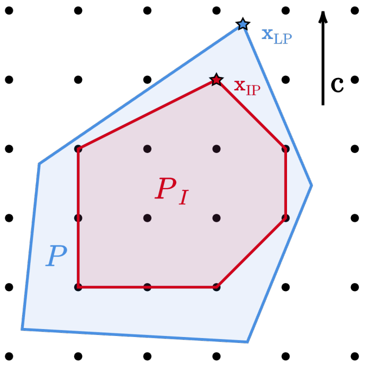
Integer Linear Programming Problems

Integer linear programming (IP) problem in standard form:

$$\begin{aligned} \max \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^n \end{aligned}$$

where $A \in \mathbb{Z}^{m\times n}, \mathbf{b} \in \mathbb{Z}^m, \mathbf{c} \in \mathbb{R}^n, m, n \in \mathbb{N}_+$.

$P := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$
$P_I := \operatorname{conv}(P \cap \mathbb{Z}^n)$



Branch-and-Bound

Select some $i \in \{1, \dots, n\}$ such that $(\mathbf{x}_{\mathrm{LP}})_i \notin \mathbb{Z}$.

$$\begin{aligned} \max \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^n \end{aligned}$$

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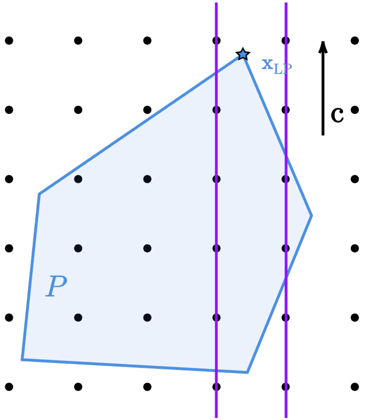

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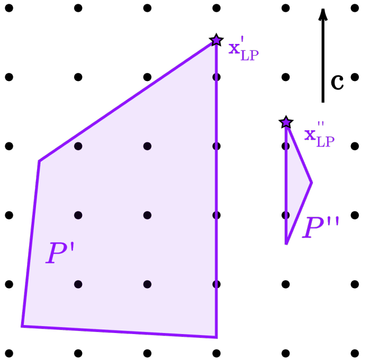

Cutting Plane

Select a halfspace $H := \{\mathbf{x} : \boldsymbol{\alpha}^\mathsf{T}\mathbf{x} \le \beta\}$ such that it contains all the integer feasible points, i.e., $P_I \subseteq H$.

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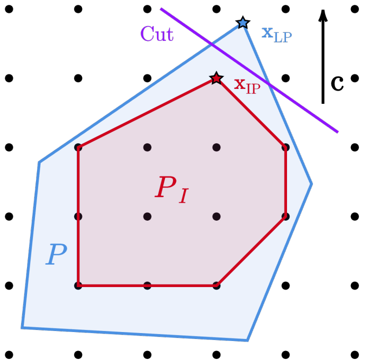

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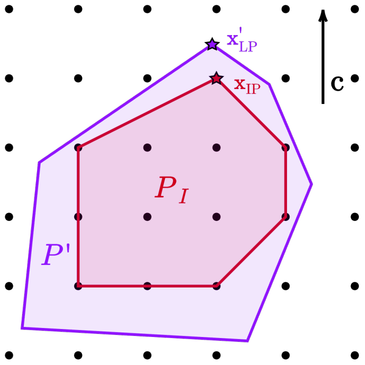
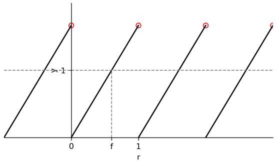
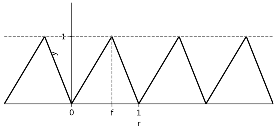


CGF Examples



(a) $CG_f(r)$



(b) $GMI_f(r)$

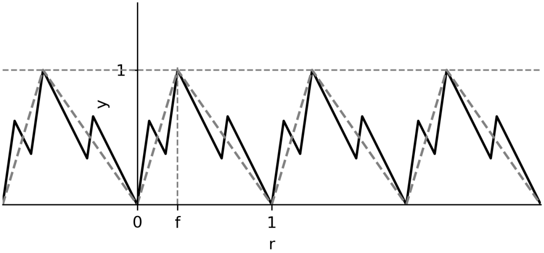
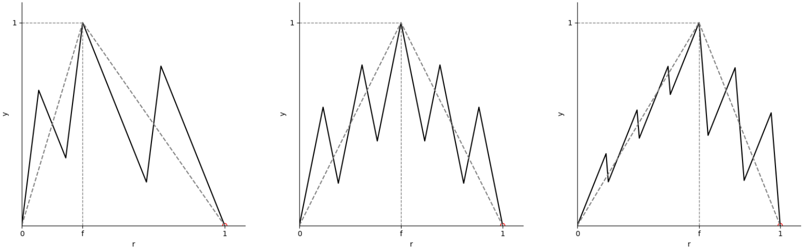


Figure: A one-dimensional cut generating function $\pi^{2,2}_{f,s_1,s_2}$

One-dimensional Extreme CGFs



(a) $\pi^{2,2}_{0.3,-2,7}(r)$ (b) $\pi^{3,3}_{0.5,-5,5}(r)$ (c) $\pi^{4,3}_{0.6,-12.5,2.5}(r)$

Figure: Three examples of the one-dimensional cut generating functions $\pi^{p,q}_{f,s_1,s_2}$ on $[0,1)$.

Learning Theory Tools

- Given $\varepsilon > 0$ and $\delta \in (0, 1)$, the following holds with probability at least $1 - \delta$ for $N \geq N(\varepsilon, \delta)$:

$$\left| \frac{1}{N} \sum_{i=1}^{N} T(I_i, \boldsymbol{\mu}) - \mathbb{E}_{I \sim \mathcal{D}} T(I, \boldsymbol{\mu}) \right| \leq \varepsilon,$$

- Pseudo-dimension can be used to bound $N(\varepsilon, \delta)$ [ABB+99]:

$$N(\varepsilon, \delta) = \mathcal{O}\left(\frac{B^2}{\varepsilon^2} \left(\mathrm{Pdim}(\{T(\cdot, \boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{P}\}) \log\left(\frac{B}{\varepsilon}\right) + \log\left(\frac{1}{\delta}\right) \right) \right),$$

Learnability Results

Theorem (Cheng and Basu)

$$\text{Pdim}\left(\left\{T(\cdot, \mathbf{s}) : \mathbf{s} \in [0,1]^2\right\}\right) = \mathcal{O}\left(n^2 \log((m+n)\varrho)\right).$$

Theorem (Cheng and Basu)

$$\text{Pdim}\left(\left\{T^k(\cdot, \boldsymbol{\mu}) : \boldsymbol{\mu} \in \Delta_k^\tau\right\}\right) = \mathcal{O}\left(kn^2 \log((m+n)\varrho) + k^2 \log(n\tau)\right).$$

Learnability Results

Use neural network to select a good CGF for each instance?

Theorem (Cheng and Basu)

$$\text{Pdim}\left(\left\{h(\cdot,\mathbf{w}) : \mathbf{w} \in \mathbb{R}^W\right\}\right) = \mathcal{O}\left(LW\log(U+2) + n^2W\log((m+n)\varrho)\right),$$

$$\text{Pdim}\left(\left\{h^k(\cdot,\mathbf{w}) : \mathbf{w} \in \mathbb{R}^W\right\}\right) = \mathcal{O}\left(LW\log(U+k) + kW\log(n\tau) + n^2W\log((m+n)\varrho)\right).$$

Numerical Experiments

Table: Average tree sizes on 100 instances, after adding a single type of cut at the root.

Problem Type	GMI	1-row cut	2-row cut	5-row cut	10-row cut	best 1-row cut
Knapsack(20, 1)	158.88	**87.0**	n/a	n/a	n/a	35.54
Knapsack(30, 1)	832.16	**58.84**	n/a	n/a	n/a	13.98
Knapsack(50, 1)	3543.91	**277.01**	n/a	n/a	n/a	125.85
Knapsack(16, 2)	399.86	316.8	**178.68**	234.09	203.66	102.07
Knapsack(30, 3)	4963.91	4311.04	3430.37	**2817.55**	2822.37	3561.36
Packing(15, 30)	389.67	**367.48**	376.86	401.87	391.44	303.72
Packing(20, 40)	1200.55	1123.9	1214.92	**1113.82**	1185.26	738.58

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References I

Martin Anthony, Peter L Bartlett, Peter L Bartlett, et al., *Neural network learning: Theoretical foundations*, vol. 9, cambridge university press Cambridge, 1999.

Maria-Florina F Balcan, Siddharth Prasad, Tuomas Sandholm, and Ellen Vitercik, *Structural analysis of branch-and-cut and the learnability of gomory mixed integer cuts*, Advances in Neural Information Processing Systems **35** (2022), 33890–33903.