Statistical Estimation in the Spiked Tensor Model via the Quantum Approximate Optimization Algorithm

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NEURAL INFORMATION PROCESSING SYSTEMS

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Spiked Tensor: a statistical model for learning tensor data

Given $$\boldsymbol{Y} = \frac{\lambda_n}{n^{q/2}}\boldsymbol{u}^{\otimes q} + \frac{1}{\sqrt{n}}\boldsymbol{W} \quad \in (\mathbb{R}^n)^{\otimes q}$$

Signal $\boldsymbol{u} \sim \mathrm{Unif}(\{\pm 1\}^n)$ *Noise* $W_{i_1,\ldots,i_q} \sim \mathcal{N}(0,1)$

Goal: (weak) recovery of the hidden signal

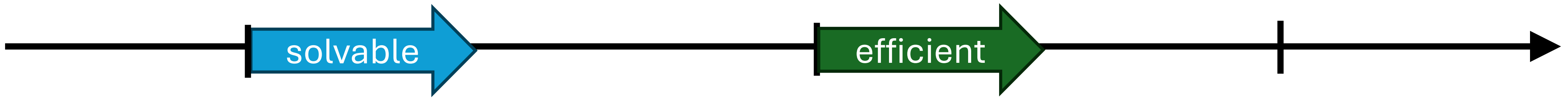
Find $\hat{\boldsymbol{u}}(\boldsymbol{Y})$ *such that* $\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n^2}\langle \hat{\boldsymbol{u}}(\boldsymbol{Y}), \boldsymbol{u}\rangle^2\right] > 0$

Spiked Tensor problem has a huge **statistical-computational gap!**

$$\boldsymbol{Y} = \frac{\lambda_n}{n^{q/2}} \boldsymbol{u}^{\otimes q} + \frac{1}{\sqrt{n}} \boldsymbol{W}$$

Signal-to-Noise Ratio (λ_n) Thresholds for Recovery

$\Theta(1)$ $\Theta\left(n^{(q-2)/4}\right)$ $\Theta\left(n^{(q-2)/2}\right)$



λ_n

Information-theoretic threshold

Classical Computational Threshold

(achieved e.g. by spectral methods)

Tensor power iteration

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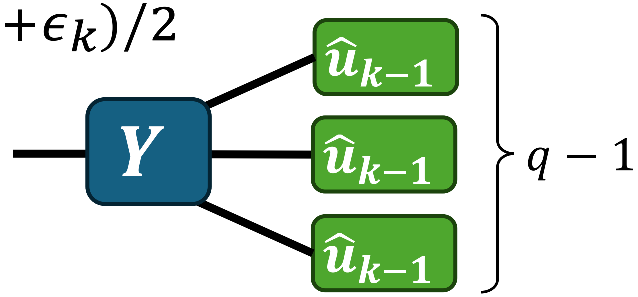
Classical algorithms for spiked tensor

- **Tensor power iteration** (k-step) works if $\lambda_n \gtrsim n^{(q-2+\epsilon_k)/2}$



$$\hat{\boldsymbol{u}}_k \propto \boldsymbol{Y}[\boldsymbol{u}_{k-1}^{\otimes q-1}] \quad \text{with} \quad \hat{\boldsymbol{u}}_0 \sim \mathrm{Unif}(\{\pm 1\}^n)$$

- **Spectral method** (with tensor unfolding) works if $\lambda_n \gtrsim n^{(q-2)/4}$

$$\mathrm{mat}(\boldsymbol{Y}) = n^{\lfloor q/2 \rfloor} \times n^{\lceil q/2 \rceil} \text{ matrix}$$

Find largest singular vector of $\mathrm{mat}(\boldsymbol{Y})$, which will have nontrivial overlap with $\mathrm{vec}(\boldsymbol{u}^{\otimes \lceil q/2 \rceil})$

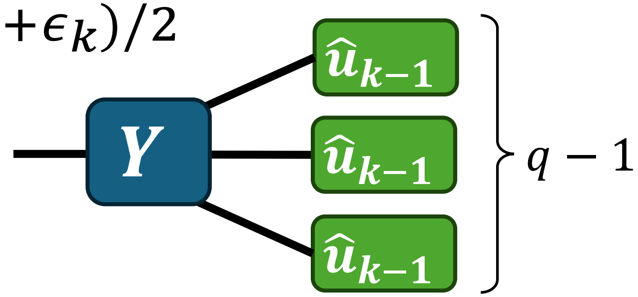
- **Maximum Likelihood Estimator** works when $\lambda_n \gtrsim \sqrt{q \log q}$

$$\hat{\boldsymbol{u}}_{\mathrm{MLE}} = \arg\max_{\boldsymbol{z}} C_{\boldsymbol{Y}}(\boldsymbol{z}) \qquad C_{\boldsymbol{Y}}(\boldsymbol{z}) = \langle \boldsymbol{Y}, \boldsymbol{z}^{\otimes q} \rangle = \sum_{i_1,\dots,i_q} Y_{i_1,\dots,i_q} z_{i_1} \cdots z_{i_q}$$

MLE is **inefficient** to compute!! NP-hard in general to find exactly

Classical algorithms for spiked tensor

- **Tensor power iteration** (k-step) works if $\lambda_n \gtrsim n^{(q-2+\epsilon_k)/2}$



$$\hat{u}_k \propto Y[u^{\otimes q-1}] \quad \text{with} \quad \hat{u}_0 \sim \mathrm{Unif}(\{\pm 1\}^n)$$

Can quantum algorithms overcome the gap?

- orks if $\lambda_n \gtrsim n^{(q-2)/4}$

Find largest singular vector of mat$(\boldsymbol{Y})$, which will have nontrivial overlap with vec$\left(\boldsymbol{u}^{\otimes \lceil q/2 \rceil}\right)$

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$$\hat{u}_{\mathrm{MLE}} = \arg\max_{z} C_{Y}(z) \qquad C_Y(z) = \langle Y, z^{\otimes q} \rangle = \sum_{i_1,\ldots,i_q} Y_{i_1,\ldots,i_q} z_{i_1} \cdots z_{i_q}$$

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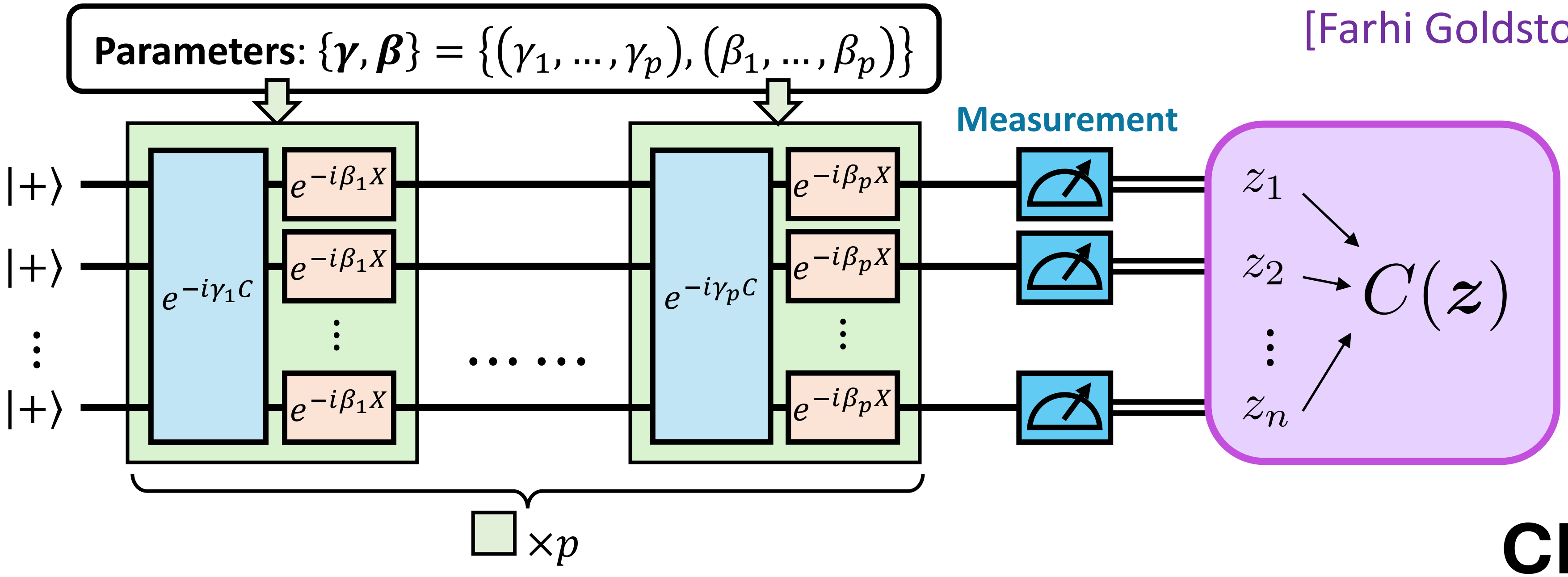
Quantum Approximate Optimization Algorithm (QAOA)

[Farhi Goldstone Gutmann 2014]



$$C\,|\boldsymbol{z}\rangle = C(\boldsymbol{z})\,|\boldsymbol{z}\rangle$$

$$B = \sum_{i=1}^{n} X_i$$

$$|\gamma, \beta\rangle = e^{-i\beta_p B} e^{-i\gamma_p C} \ldots e^{-i\beta_1 B} e^{-i\gamma_1 C} \, |+\rangle^{\otimes n}$$

Choose γ, β to maximize $\langle C \rangle$

- Simple & easy implementation (e.g. ions (2019), superconductors (2020) and atoms (2022))
- Cannot classically sample (“supremacy”) even @ p=1 [Farhi Harrow ’16] [Krovi ’22]
- ***Guaranteed to get $C_{\max}$ as depth $p \to \infty$!***

Our Result: QAOA for Spiked Tensor

- Apply QAOA to approximately maximize $C_{\boldsymbol{Y}}(\boldsymbol{z}) = \langle \boldsymbol{Y}, \boldsymbol{z}^{\otimes q} \rangle$

- Suppose $\lim_{n\to\infty} \frac{\lambda_n}{n^{(q-2+\varepsilon_p)/2}} = \Lambda$ $\qquad \varepsilon_p = \begin{cases} \frac{q-2}{(q-1)^p-1}, & q > 2, \\ 1/p, & q = 2. \end{cases}$

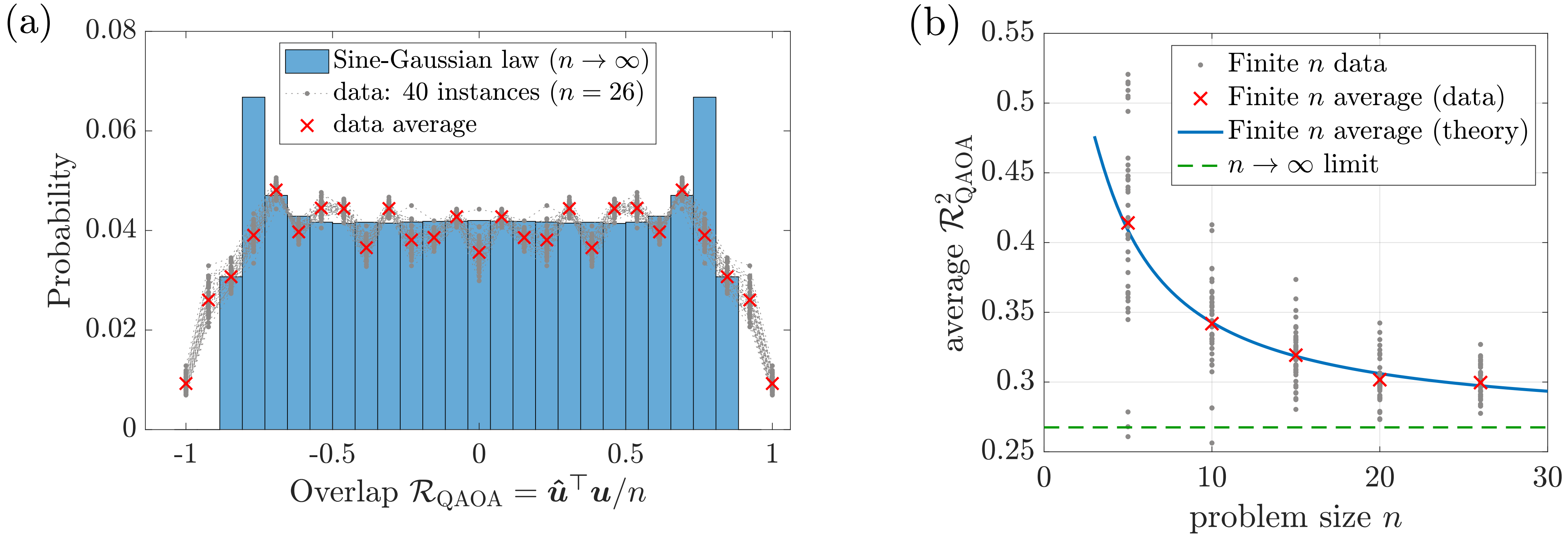
- Let $\hat{\boldsymbol{u}} \sim |\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle$ and $R_{QAOA} = \hat{\boldsymbol{u}}^T \boldsymbol{u}/n$ be its overlap with the signal, then

$$\mathcal{R}_{\mathrm{QAOA}} \xrightarrow{d} a_p \sin\left[b_p \Lambda^{1/\varepsilon_p} G^{(q-1)^p}\right] \qquad G \sim \mathcal{N}(0,1)$$

$a_p(\boldsymbol{\gamma}, \boldsymbol{\beta})$, $b_p(\boldsymbol{\gamma}, \boldsymbol{\beta})$ can be calculated classically in $\tilde{O}(4^p)$ time

QAOA for Spiked Tensor: a Sine-Gaussian Law

Example numerical simulation at ***p*=1, *q*=2**
(1-step QAOA for spiked matrix)



Comparing QAOA and Power Iteration

Let $$\lim_{n\to\infty} \frac{\lambda_n}{n^{(q-2+\varepsilon_p)/2}} = \Lambda$$

$$\varepsilon_p = \begin{cases} \frac{q-2}{(q-1)^p-1}, & q > 2, \\ 1/p, & q = 2. \end{cases}$$

Note:
$1/\epsilon_p \sim \exp(p)$
when $q > 2$

p-step QAOA

$$\mathcal{R}_{\text{QAOA}} \xrightarrow{d} a_p \sin\left[b_p \Lambda^{1/\varepsilon_p} G^{(q-1)^p}\right]$$

$$\approx (|a_p b_p|^{\varepsilon_p} \Lambda)^{1/\varepsilon_p} G^{(q-1)^p}$$

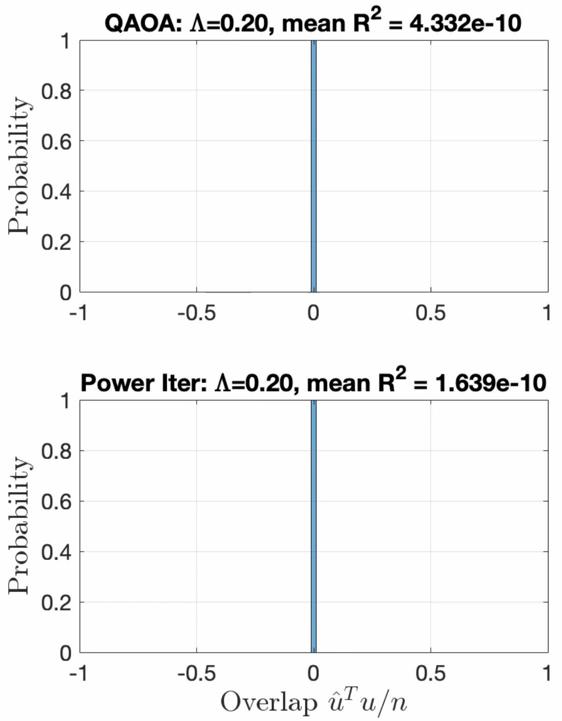
$G \sim \mathcal{N}(0,1)$

quantum enhancement factor

p-step Power Iteration

$$\mathcal{R}_{\text{PI}} \xrightarrow{d} \sin\left[\tan^{-1}\left(\Lambda^{1/\varepsilon_p} G^{(q-1)^p}\right)\right]$$

$$\approx \Lambda^{1/\varepsilon_p} G^{(q-1)^p}$$



QAOA effectively improves the recovery threshold vs. power iteration

Quantum enhancement factor $|a_p b_p|^{\varepsilon_p}$

p \ q	2	3	4	5	6	7
1	0.8578	1.0505	1.2131	1.3562	1.4857	1.6047
2	0.9663	1.0505	1.1916	1.2882	1.4167	1.5162
3	1.0204	1.0314	1.1615	1.2555	1.3844	1.4917
4	1.0487	1.0144	1.1419	1.2447	1.3795	1.4858
5	1.0631	1.0063	1.1327	1.2411	1.3770	1.4845
6	1.0697	1.0013	1.1297	1.2399	1.3743	1.4842
7	1.0719					

Effective recovery thresholds:

p-step QAOA: $\lambda_n \simeq |a_p b_p|^{-\epsilon_p} n^{(q-2+\epsilon_p)/2}$

p-step Power iteration: $\lambda_n \simeq n^{(q-2+\epsilon_p)/2}$

Classical computational threshold: $\lambda_n \propto n^{(q-2)/4}$

p-step QAOA with tensor unfolding: $\lambda_n \simeq |a_p b_p|^{-\epsilon_p} n^{(q-2+q/p)/4}$

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Quantum enhancement factor

$|a_p b_p|^{\varepsilon_p}$

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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.4917
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.4845
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.4842

For more details, see

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Wednesday 11am – 2pm

December 11

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