Reconsidering Overfitting in the Age of Overparameterized Models

slides & refs

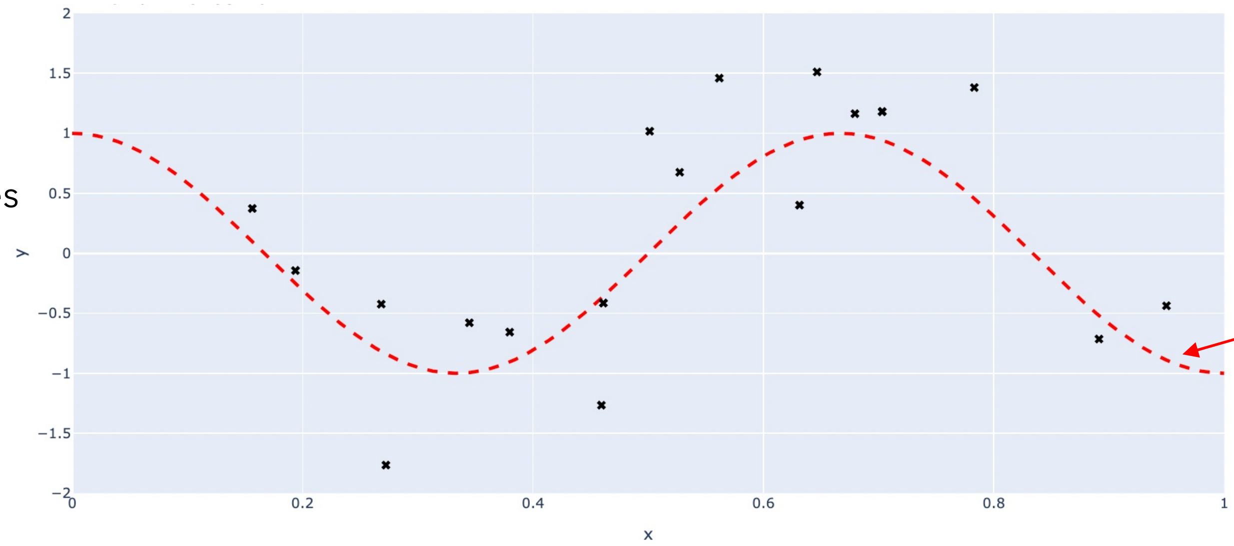
NeurIPS 2023 Tutorial, New Orleans

Speakers: Spencer Frei, Vidya Muthukumar, Fanny Yang, Moderator: Daniel Hsu

UC DAVIS UNIVERSITY OF CALIFORNIA · Georgia Tech · ETH zürich · COLUMBIA UNIVERSITY

Textbook wisdom on overfitting

n = 20 samples



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true $f^\star(x)$

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polynomial fit
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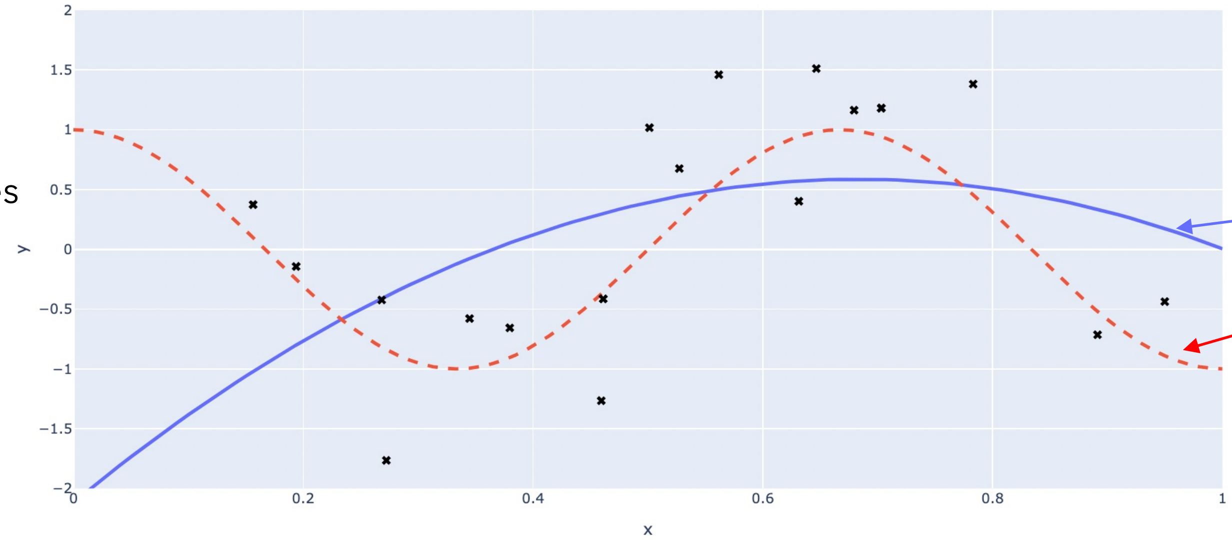

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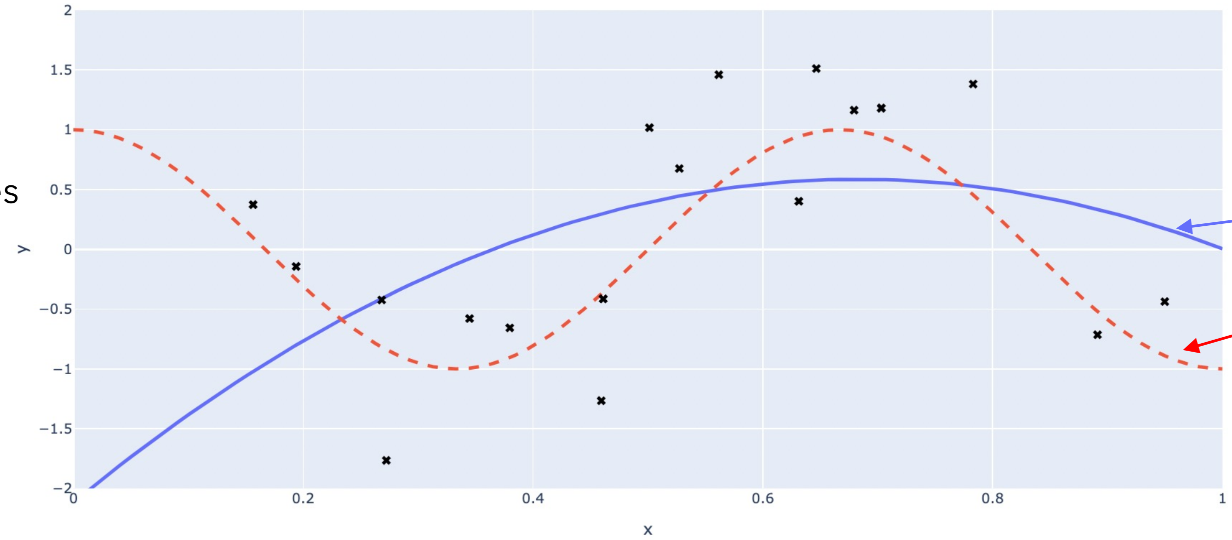

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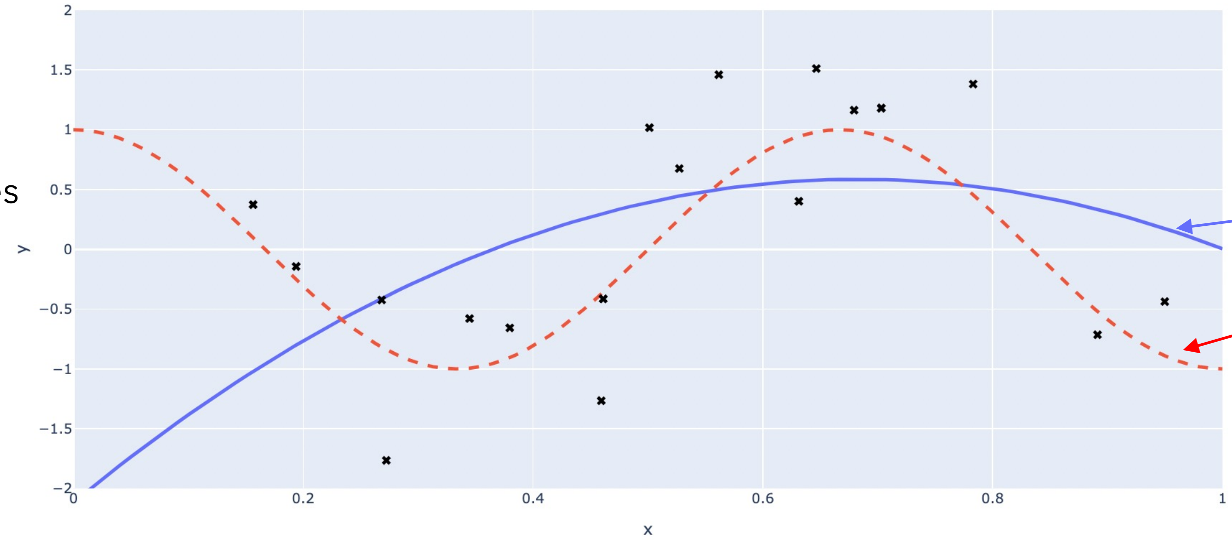

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Small models cannot fit perfectly:

- cannot express function of interest (high statistical bias)
- largely ignores noise → does not fluctuate a lot (small variance)

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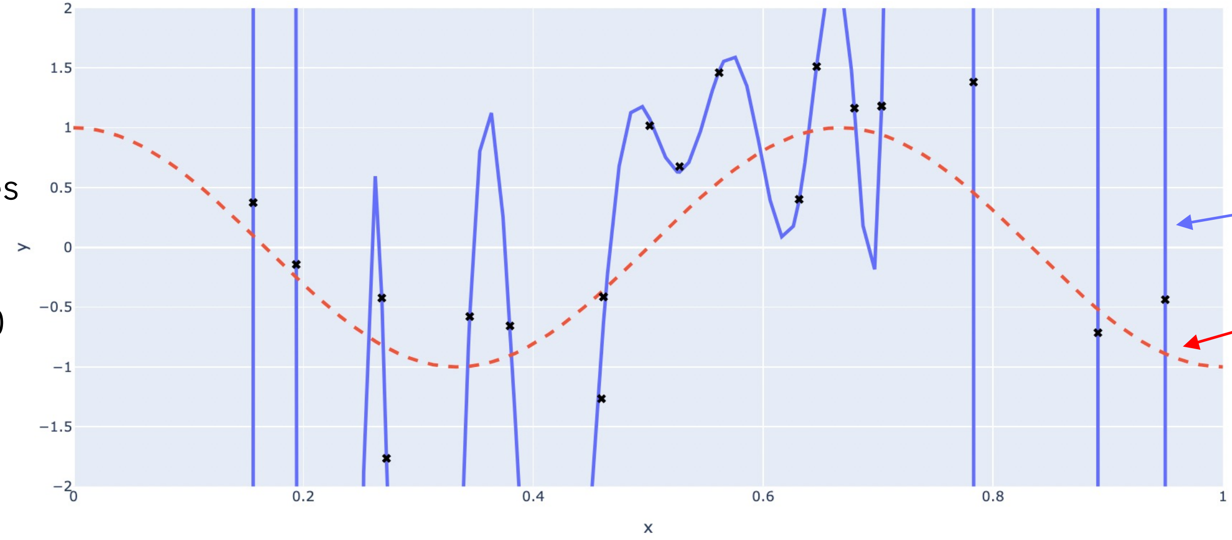

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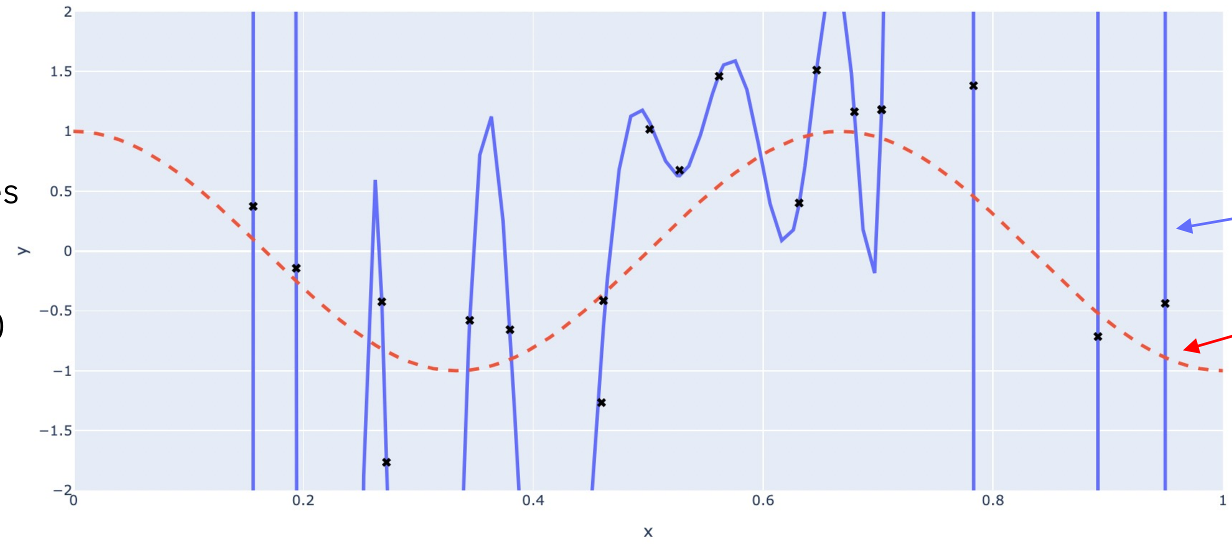

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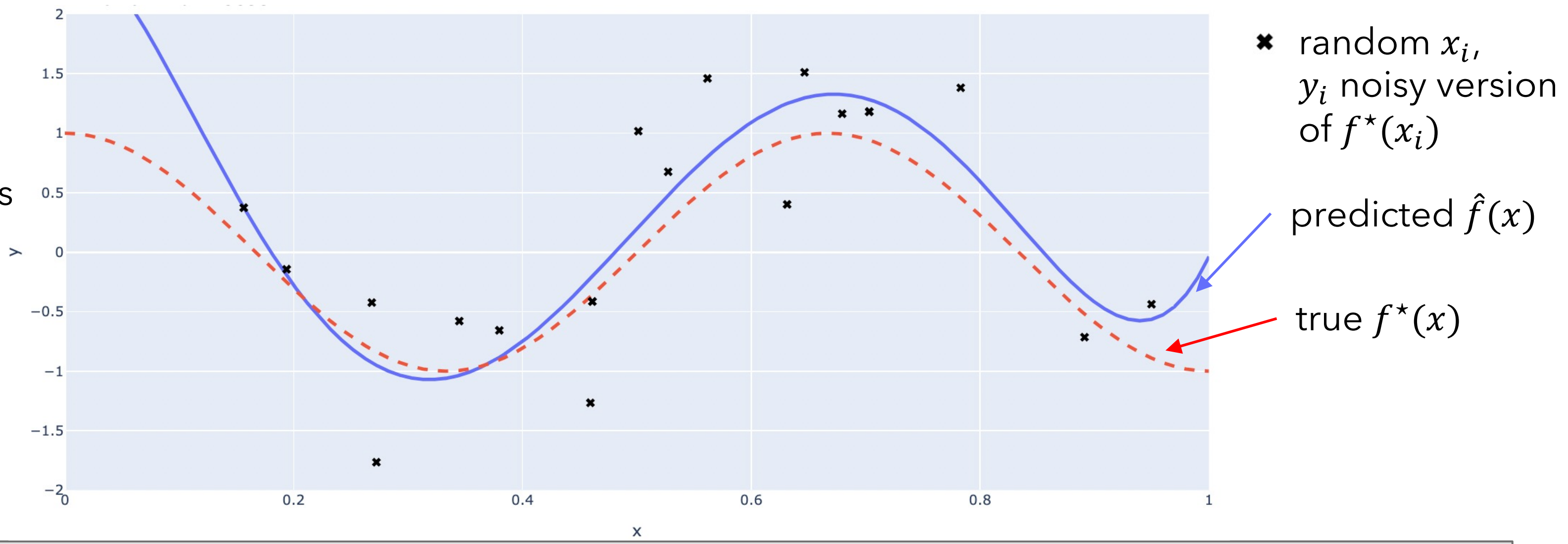
Large models fit perfectly (overfit):

- flexible and can express function of interest (small bias)
- fits too much of the noise (overfit) → fluctuates a lot (high variance)

Textbook wisdom: Avoid fitting noise

n = 20 samples

polynomial fit
degree d = 5

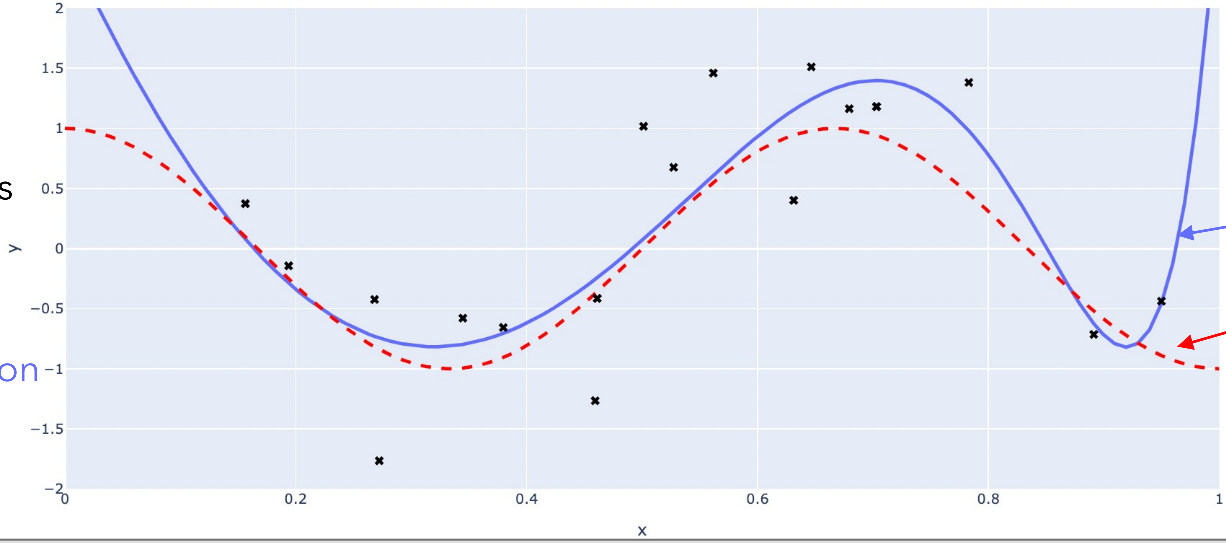


Classical theory: Improve generalization by optimizing expressivity via bias-variance trade-off

Textbook wisdom: Avoid fitting noise

n = 20 samples

polynomial fit
degree d = 20
w/ regularization



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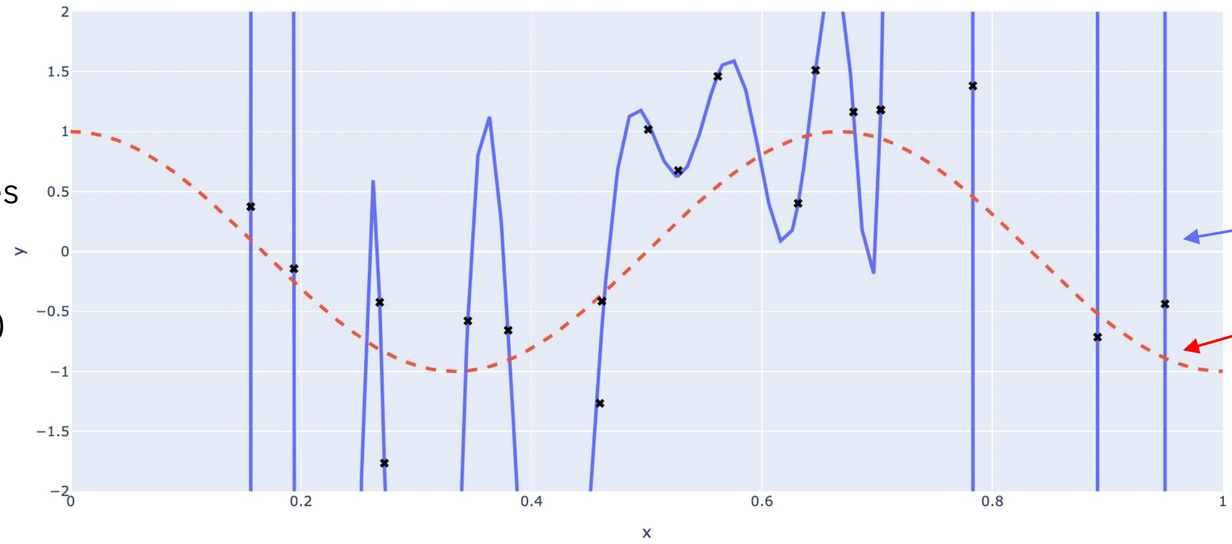

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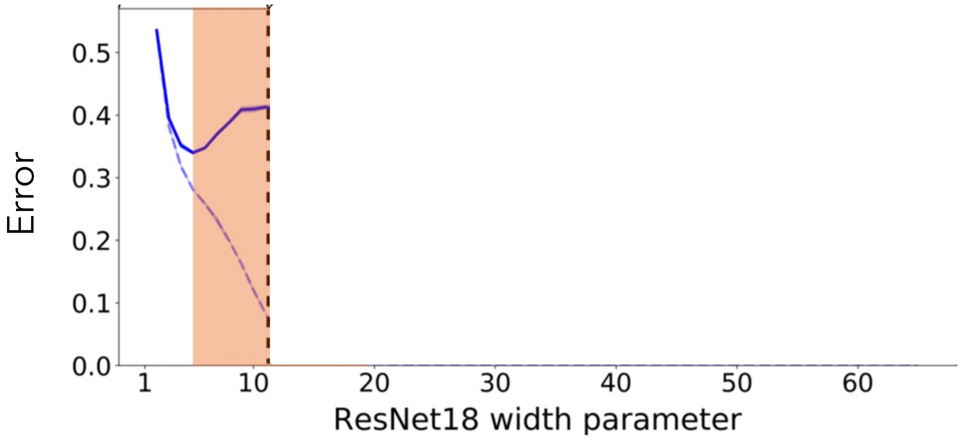
What happens if we increase the polynomial degree even further without regularizing?

Double descent on neural networks

Classification using neural networks and Adam on CIFAR-10 *with 15% additional label noise*



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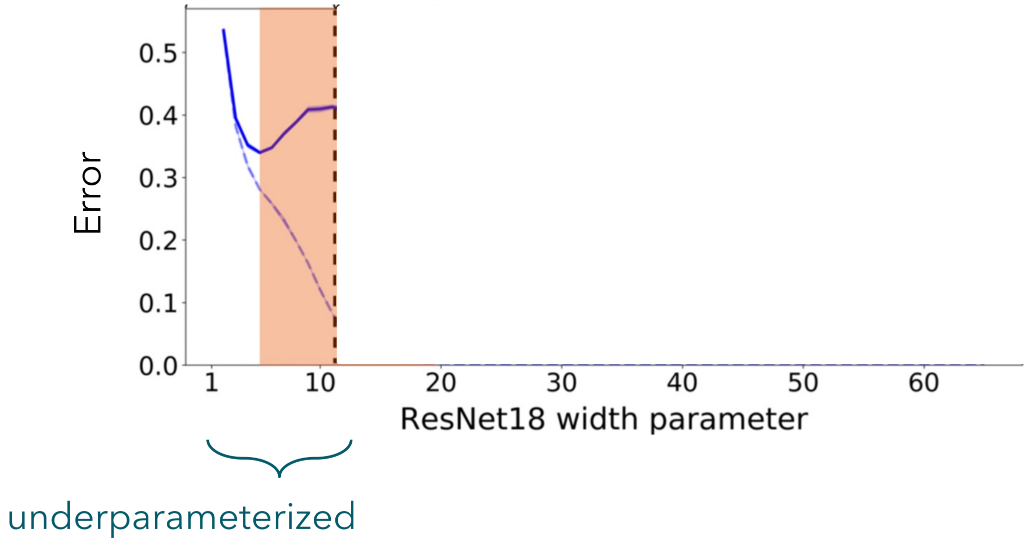

[Nakkiran, Kaplun, Bansal, Yang, Barak, Sutskever '20]

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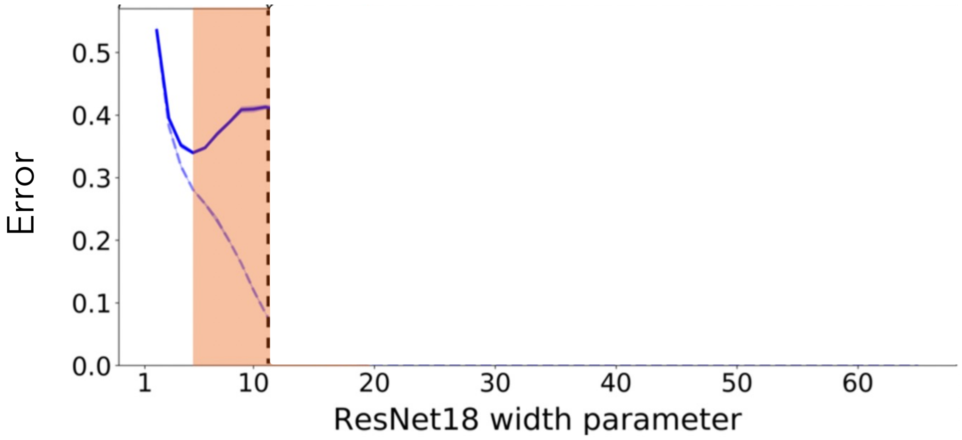

underparameterized

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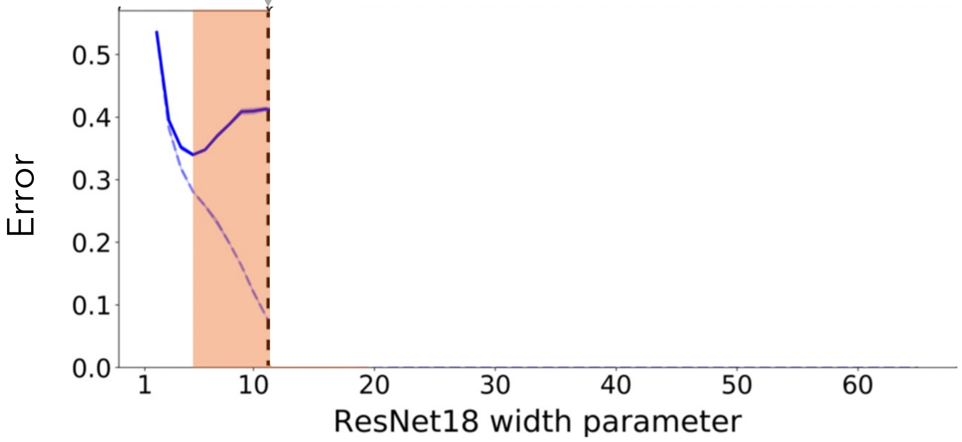
training 0-1 error ≈ 0



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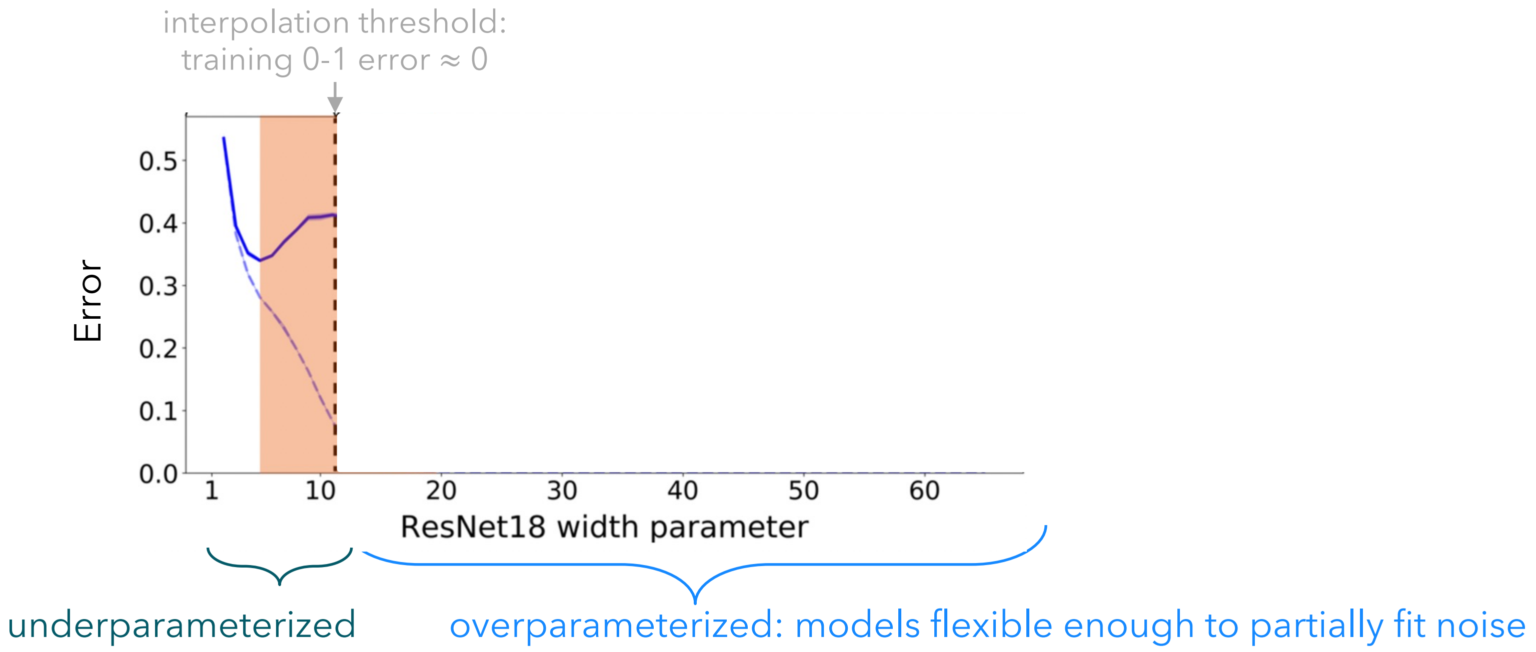

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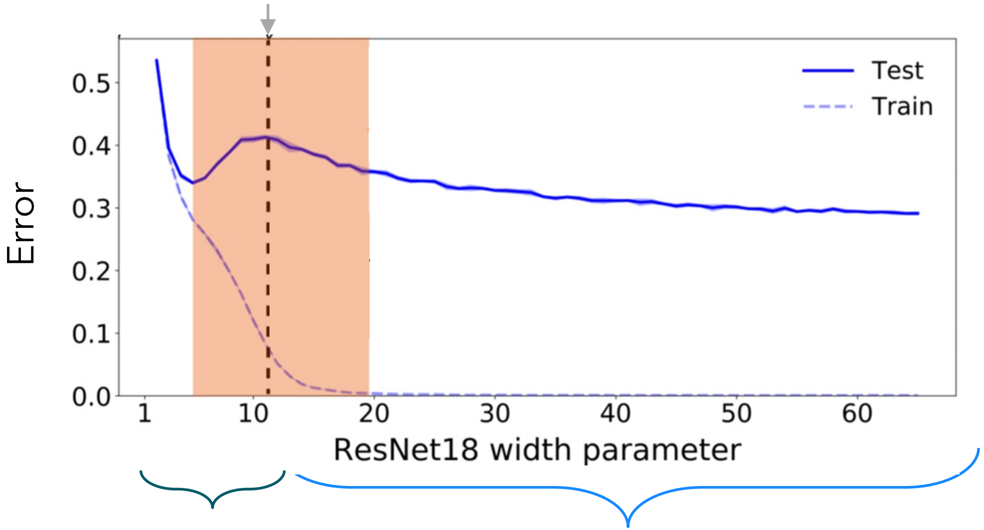


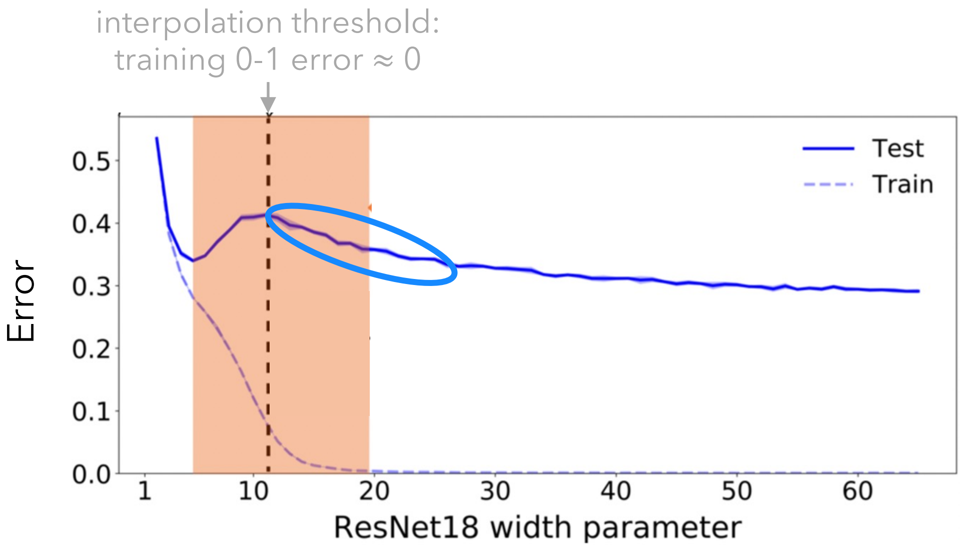
underparameterized

overparameterized: models flexible enough to partially fit noise

[Nakkiran, Kaplun, Bansal, Yang, Barak, Sutskever '20]

Obs. I: Second descent beyond interpolation

Classification using neural networks and Adam on CIFAR-10 *with 15% additional label noise*



1 After interpolation threshold, we have a second "descent" (double descent) for interpolators

[Nakkiran, Kaplun, Bansal, Yang, Barak, Sutskever '20]

Obs. II: Harmless interpolation for large models

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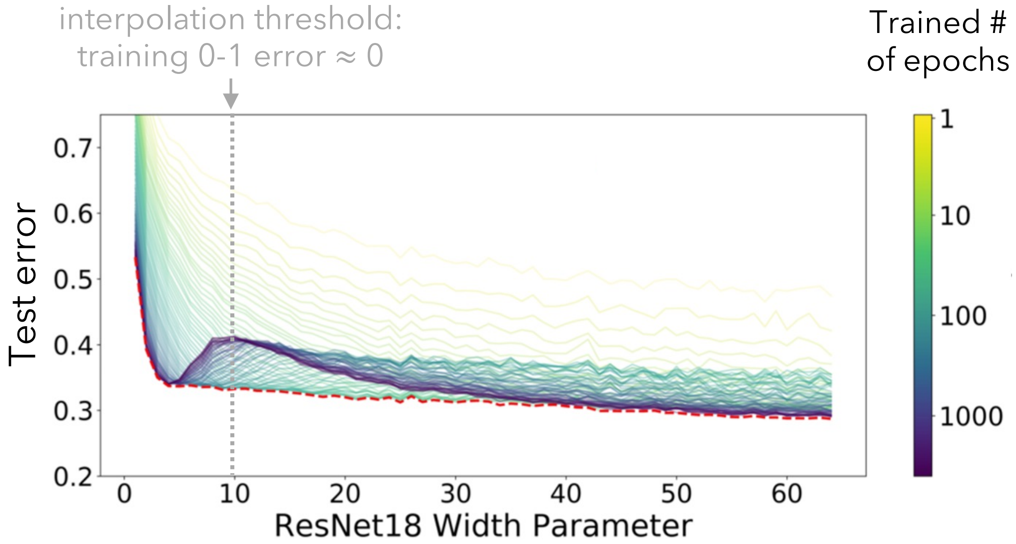

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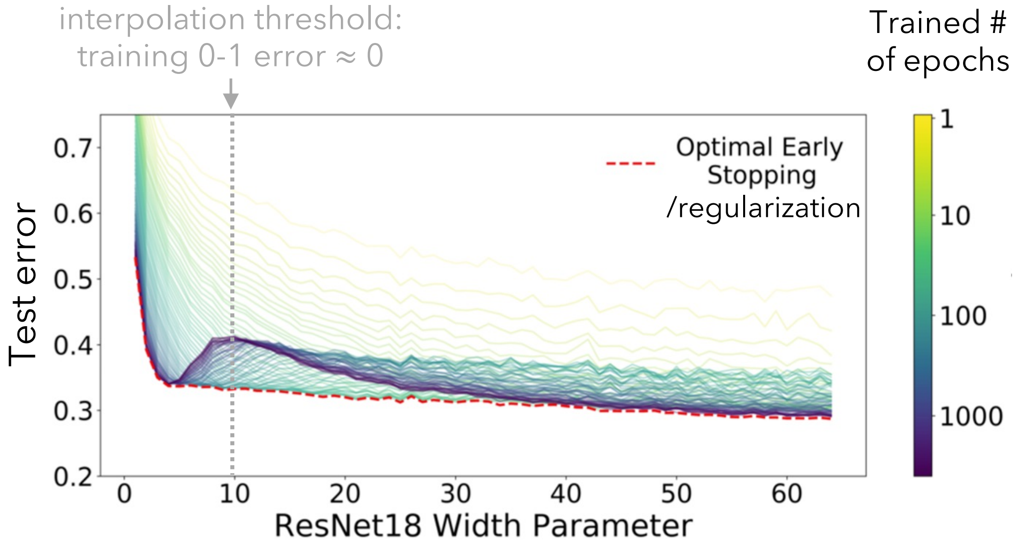

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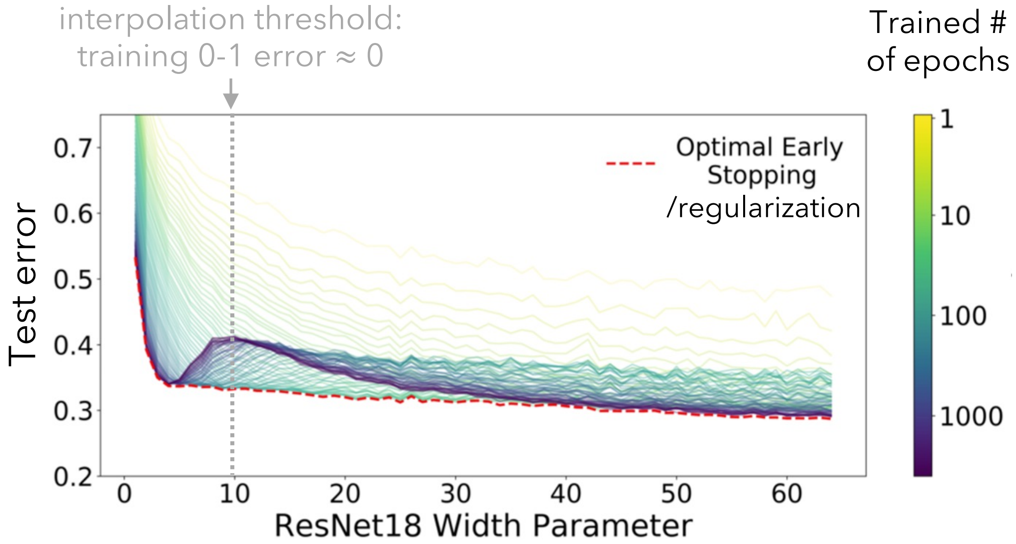

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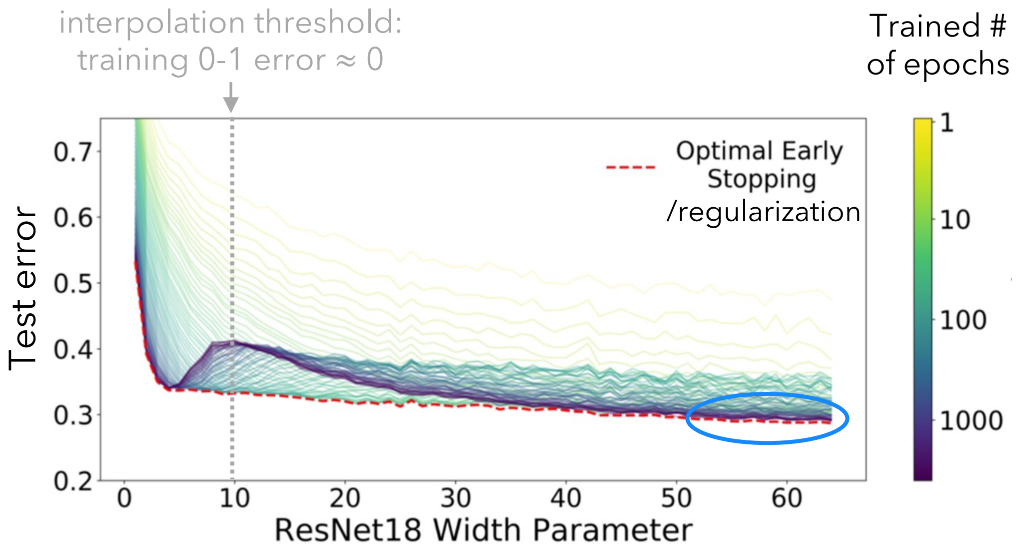

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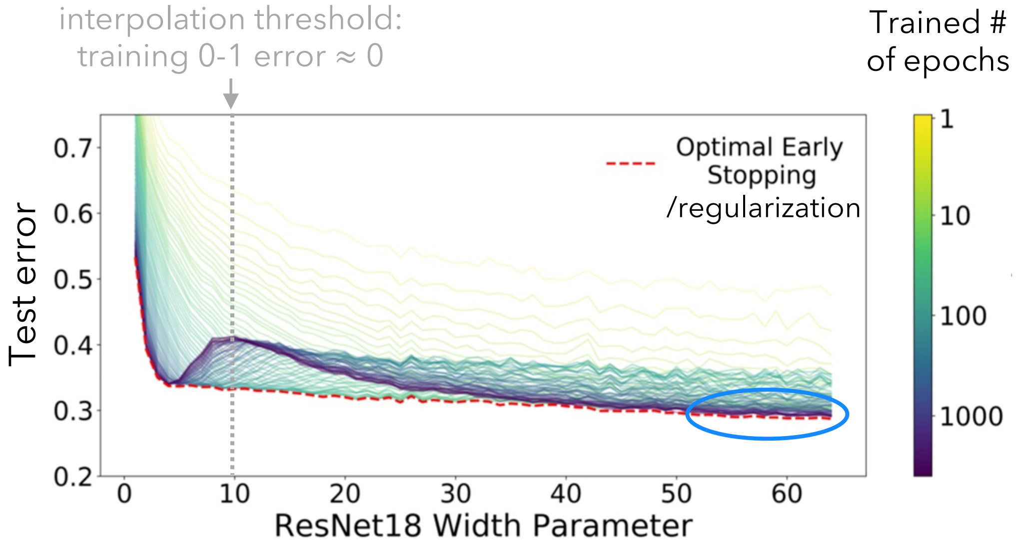

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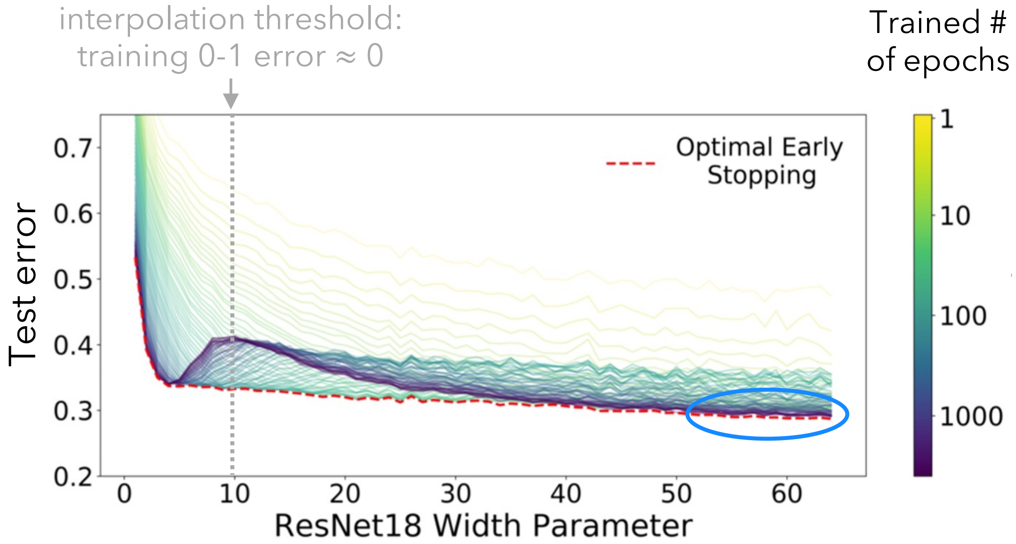
2 For large models, interpolation is not worse than regularization (harmless interpolation)

[Nakkiran, Kaplun, Bansal, Yang, Barak, Sutskever '20]

Obs. III: Good generalization for large models

Classification using neural networks and Adam on CIFAR-10 *with 15% additional label noise*



3 For large models, we achieve reasonably good test accuracy

[Nakkiran, Kaplun, Bansal, Yang, Barak, Sutskever '20]

Textbooks need an update…

uploaded 2016

DOI:10.1145/3446776

Check for updates

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panelist today

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*and many more papers that expressed the need for "rethinking"

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Try to understand when the following happens:

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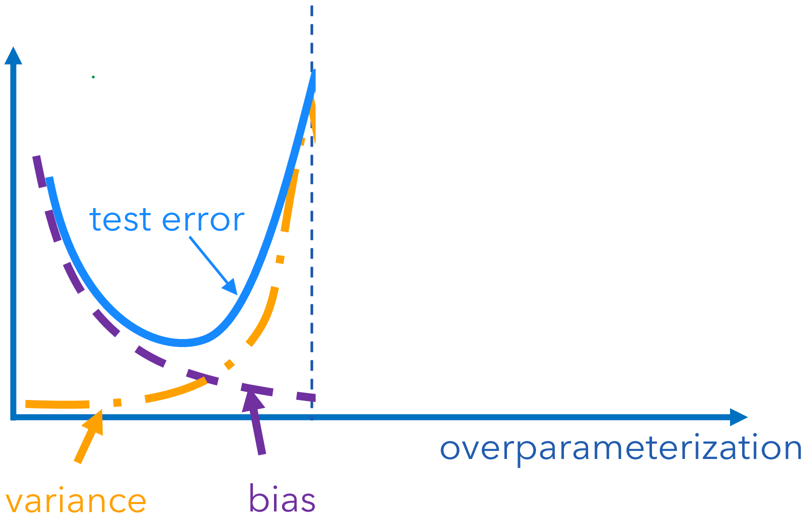

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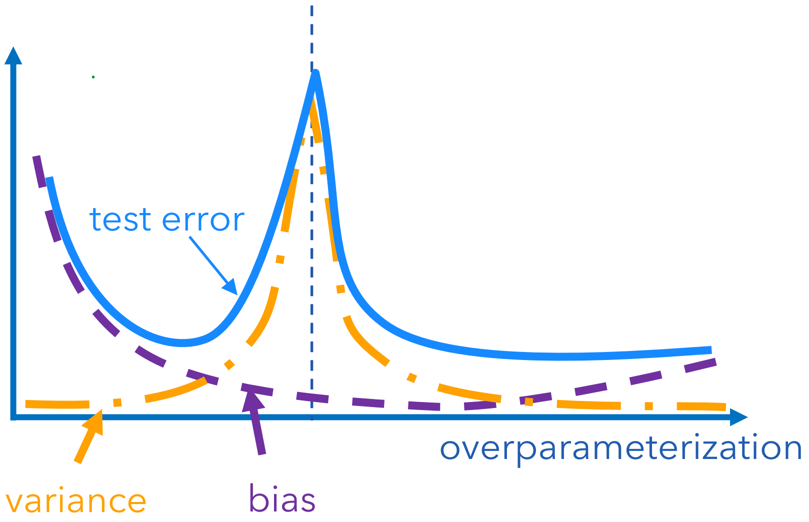

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← variance decays (1, 2)

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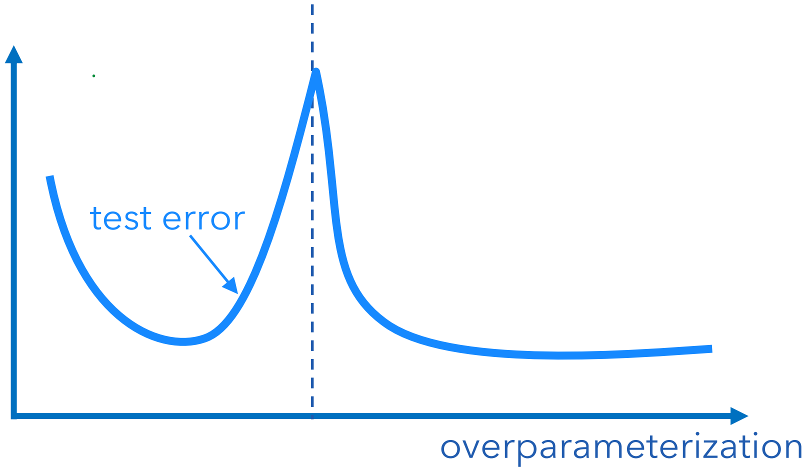
when is this the case?

Which factors govern…

when we have this picture…



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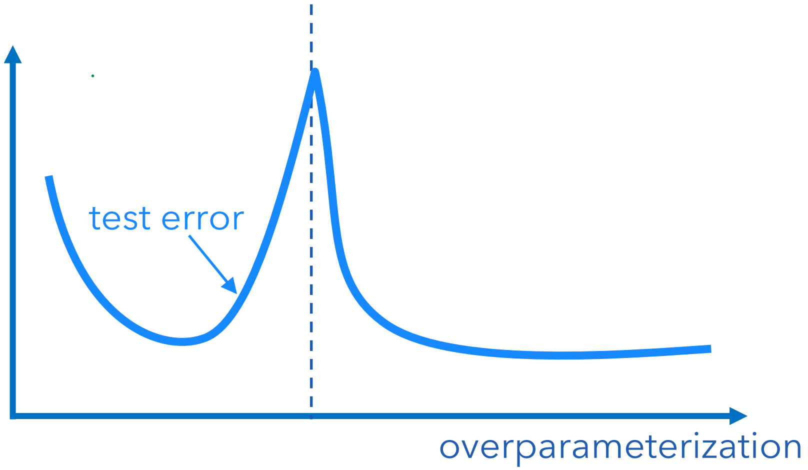

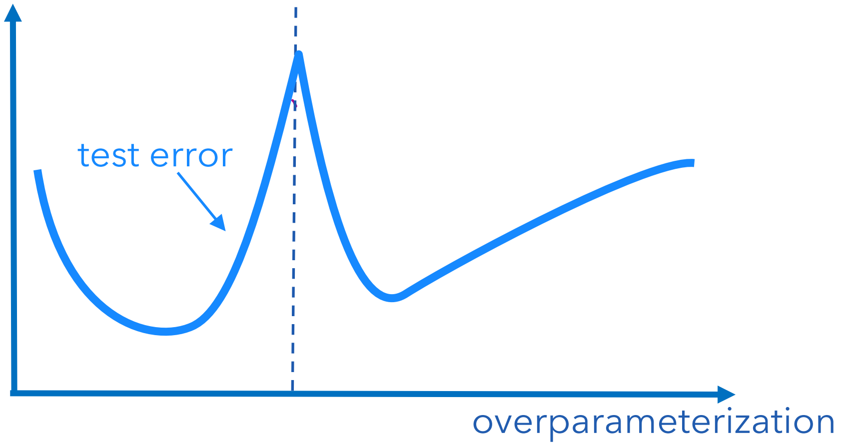

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Seeking answers using theoretical analysis…

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← complexity to analyze model

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Part II: For classification, we discuss the

- effect of loss function choices
- implicit bias of optimization algorithms for neural networks
- generalization of neural networks on noisy, high-dimensional data

*Goal is **not to find** better interpolators in practice*

*but **to understand when** interpolation is benign*

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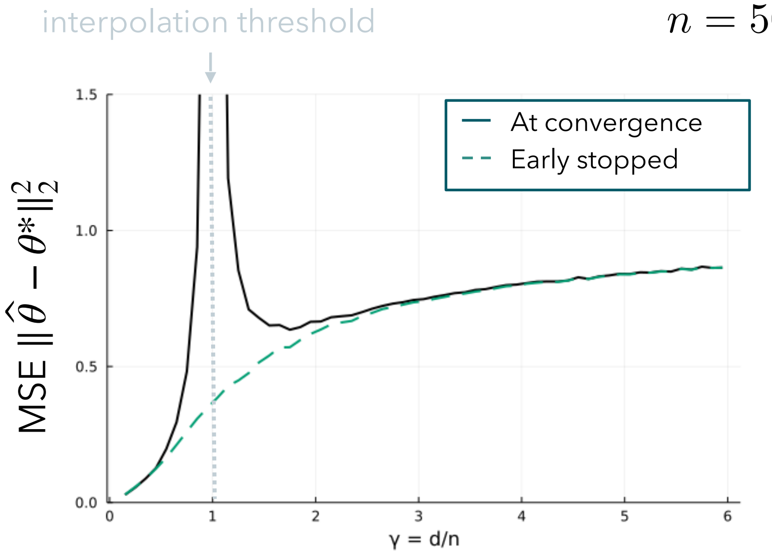
$n = 500, \theta^* = \hat{e}_1, \sigma^2 = 0.25$



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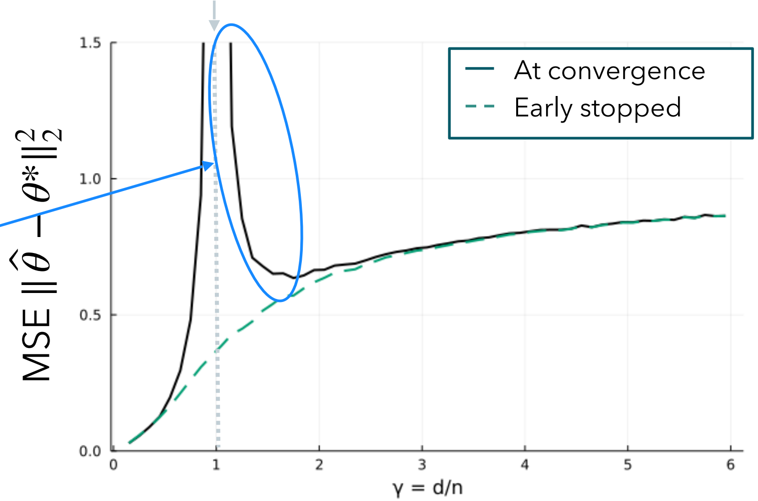

1

Second Descent after interpolation

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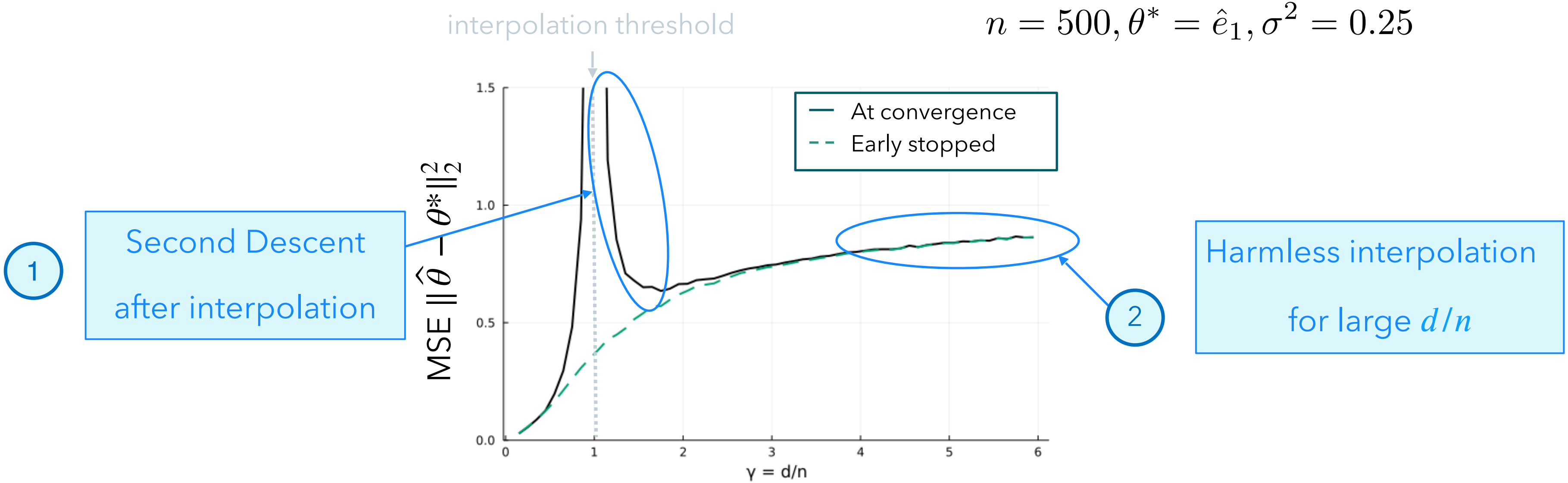

Formal setup: overparameterized linear regression

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$$Y = X^\top \theta^* + W$$

- θ^*: true parameter/signal (unknown)
- Y: output
- X: input features, dimension = d
- W: noise, variance = σ^2

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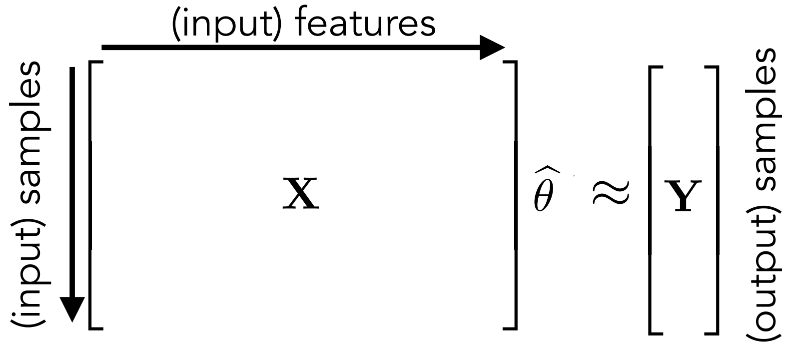

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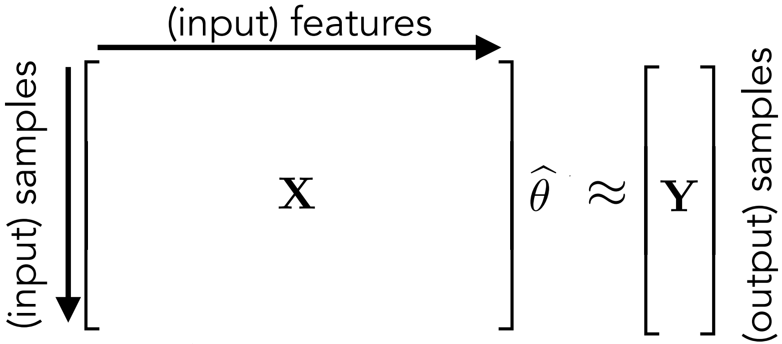

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The minimum-lp-norm interpolator

$$\widehat{\theta}_p = \arg\min \|\theta\|_p \text{ subject to } \mathbf{X}\theta = \mathbf{Y}.$$

(beginning with p = 2)

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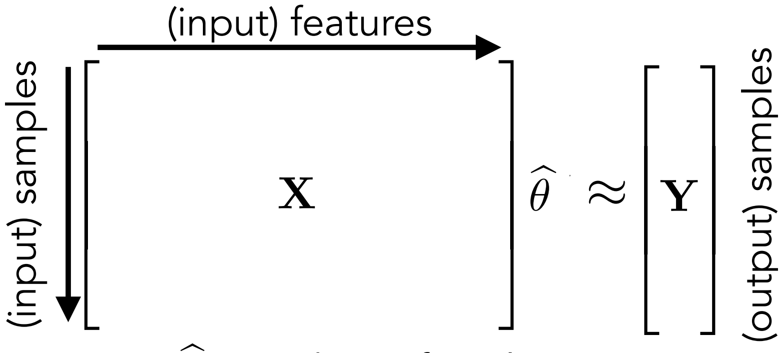

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Error metric is **mean-squared-error:** $\mathscr{E}_{\text{MSE}} := \mathbb{E}\left[(X^\top(\widehat{\theta} - \theta^*))^2\right]$

Analysis framework

Non-asymptotic: we consider $d = n^{\beta}, \beta > 1$ (or $d \gg n$) and state results as:

- **Consistency:** goal is to have $\mathscr{E}_{\text{MSE}} \to 0$ as $n \to \infty$
- **Rates:** upper and lower bounds on $\mathscr{E}_{\text{MSE}}$ as a function of n that match up to universal constants (not depending on n, d, θ^*, Σ)

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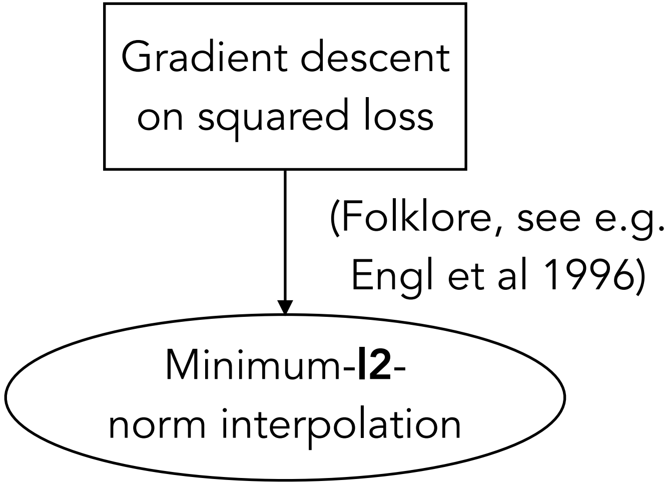
An alternative asymptotic analysis framework (not the focus of this tutorial):

Considers $d \propto n, \dfrac{d}{n} = \gamma$.

Exact error expressions derived as a function of γ as $n, d \to \infty$ together.

Why these types of “low-norm” interpolators?

Popular optimization algorithms converge to “low-norm” solutions!



$$\widehat{\theta}_2 = \arg\min \|\theta\|_2 \quad \text{subject to} \quad X_i^\top \theta = Y_i, i \in [n].$$

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Mirror descent on squared loss,
Potential = $\|\cdot\|_p$

(Gunasekar et al, 2018)

Minimum-**lp**-norm interpolation

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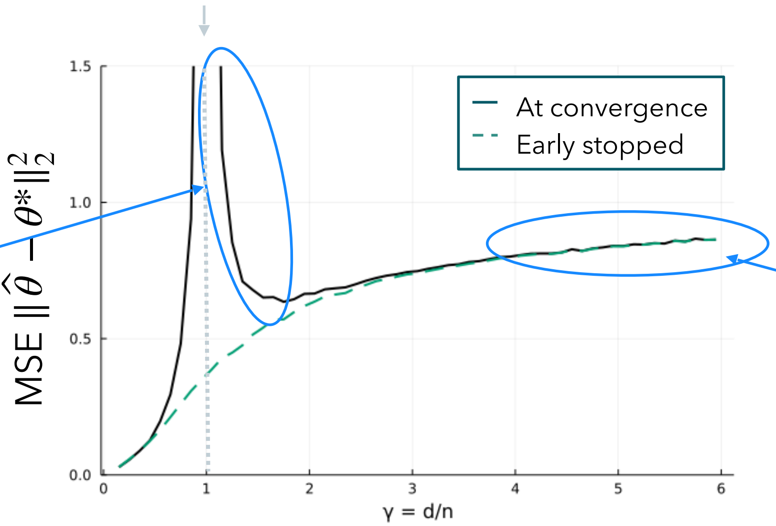
Implicit bias theory is a useful "sanity check" but not the full picture: do these solutions always generalize well?

Recall: what was observed for min-l2-norm interpolator

interpolation threshold

$$n = 500, \theta^* = \hat{e}_1, \sigma^2 = 0.25$$



1 Second Descent after interpolation

2 Harmless interpolation for large d/n

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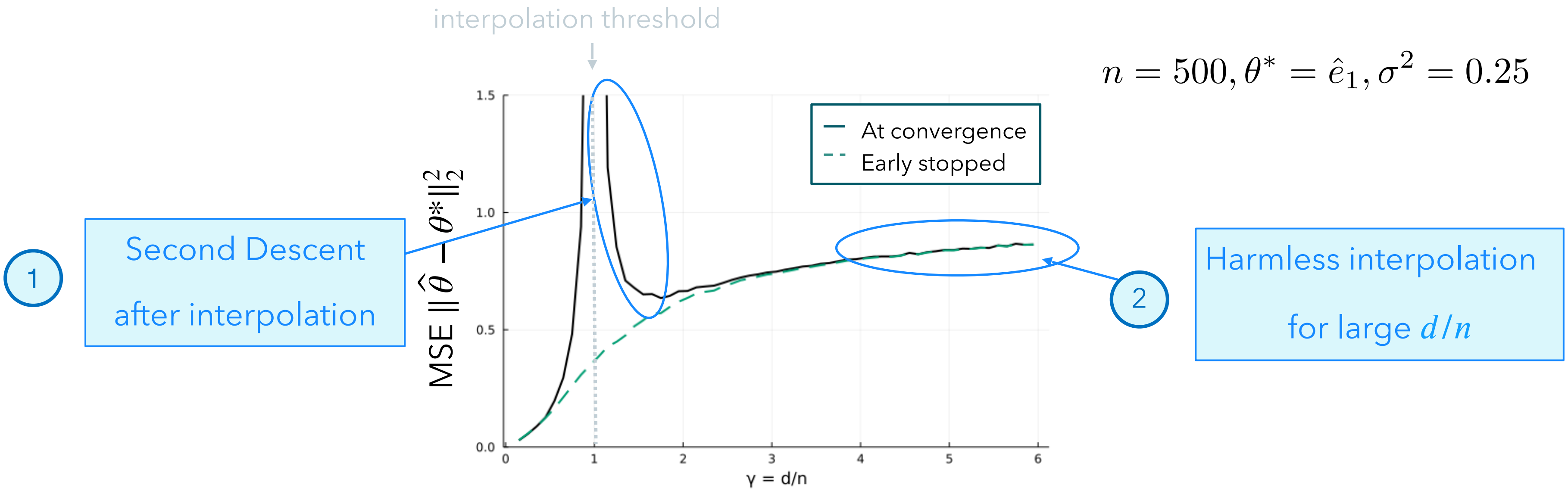

(1) and (2) are implied by **variance reduction with increased overparameterization!**

Theorem (isotropic covariance)*: $\text{Variance term} \asymp \frac{\sigma^2 n}{d}$.

*included in results of Hastie et al (2022), Bartlett et al (2020), Muthukumar et al (2020)

Plan today…

Part I: For linear regression, we discuss how

- variance can decay as overparameterization increases (simple math)
- Two factors can govern variance decay vs. bias increase
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Variance reduction: main proof ideas

- **Step 1:** minimum-l2-norm interpolator can be expressed in closed form

$$\widehat{\theta}_2 = \mathbf{X}^\top(\mathbf{X}\mathbf{X}^\top)^{-1}\mathbf{Y} = \mathbf{X}^\top(\mathbf{X}\mathbf{X}^\top)^{-1}\mathbf{X}\theta^* + \mathbf{X}^\top(\mathbf{X}\mathbf{X}^\top)^{-1}\mathbf{W}$$

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- **Step 2:** variance term can also be expressed in closed form

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Note: this calculation is simplified for isotropic data covariance, but works more generally (Bartlett et al, 2020)

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Variance reduction: main proof ideas

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Intuition: noise energy is "spread out" along d feature dimensions, contributes more harmlessly as d increases

Note: can show corresponding **precise** results when $d \propto n$, $d, n \to \infty$ (Hastie et al, 2022)

So is min-l2-norm interpolation *always* a good idea?

Interpolator $\widehat{\theta}_2 = \arg\min \|\theta\|_2$ subject to $\mathbf{X}\theta = \mathbf{Y}$ vs.

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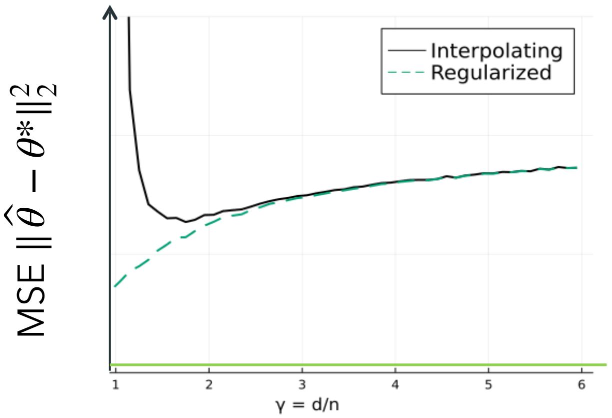
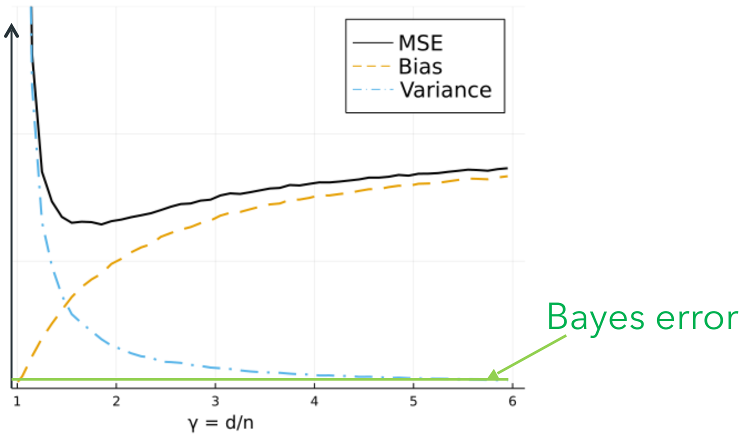
$n = 500, \theta^* = \hat{e}_1, \sigma^2 = 0.25$





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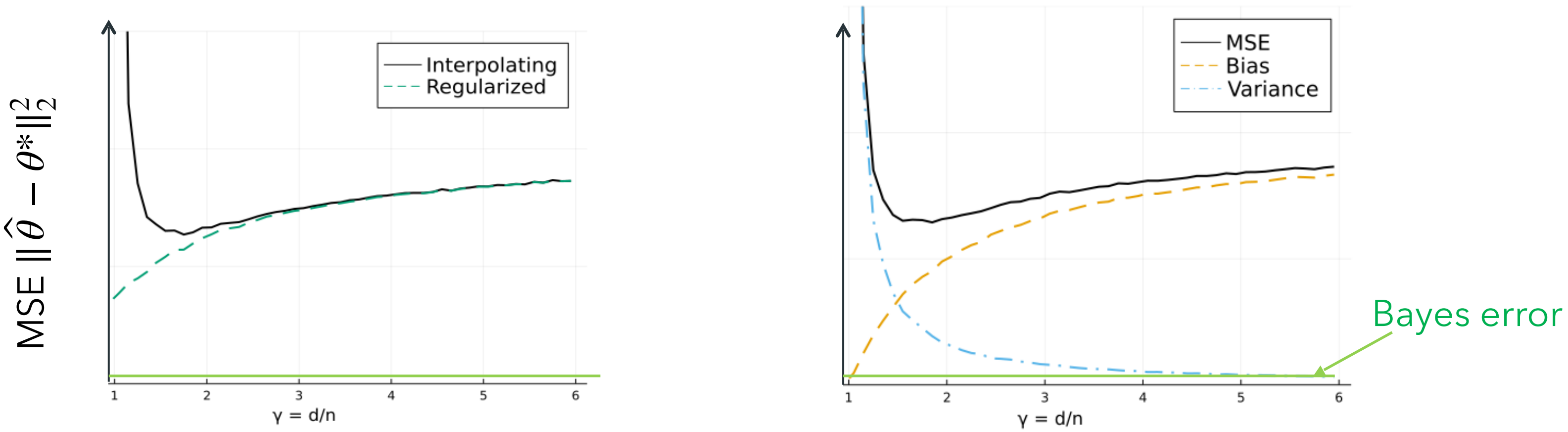

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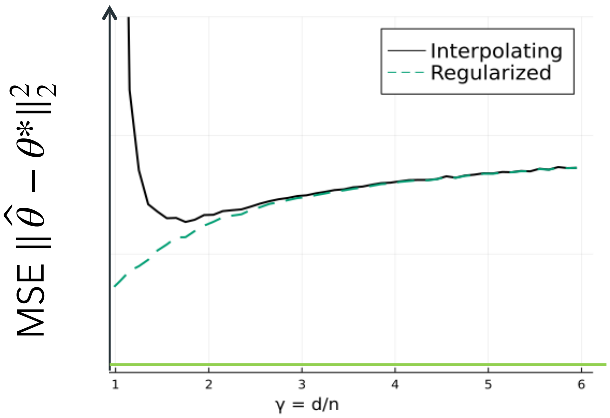

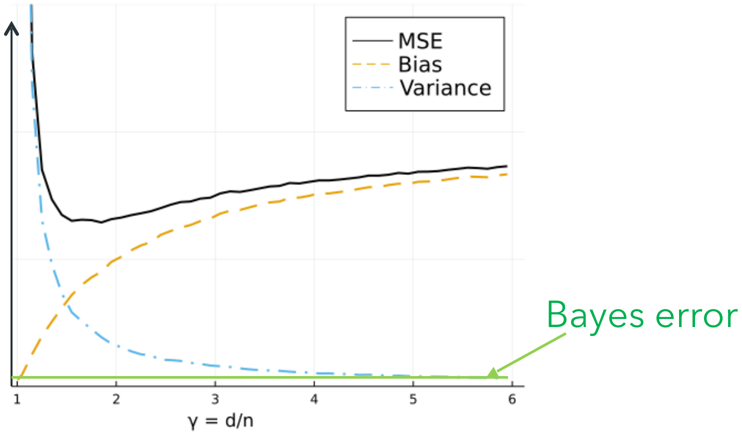

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Core issue: **bias increases with d,** eventually dominates

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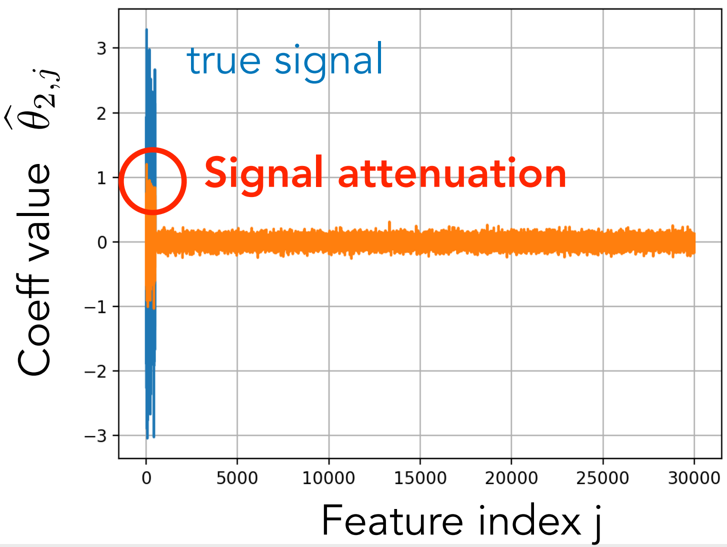

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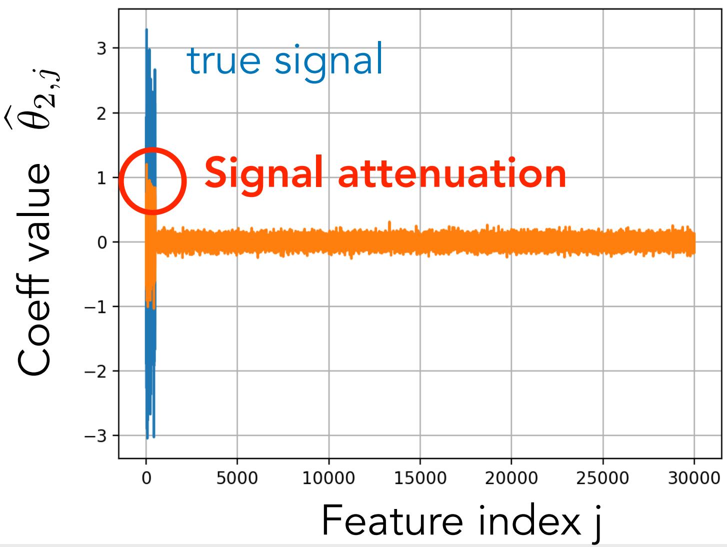

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Anisotropy to the rescue: "upweighting" features aligned with signal

- A special case $\Sigma = \begin{bmatrix} R\mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{d-k} \end{bmatrix}, R \gg 1$ (spiked-covariance)

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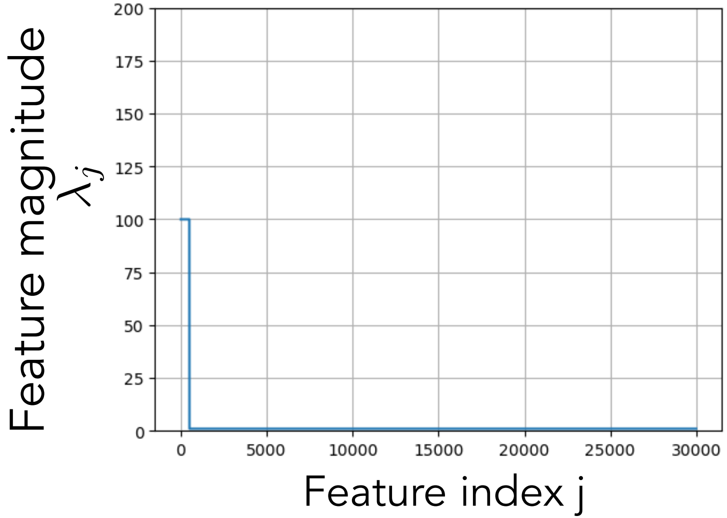
Effective "upweighting" on top k features



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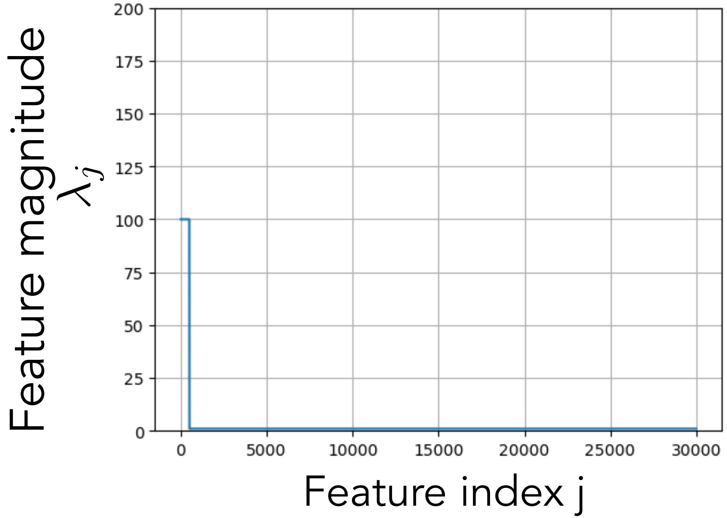
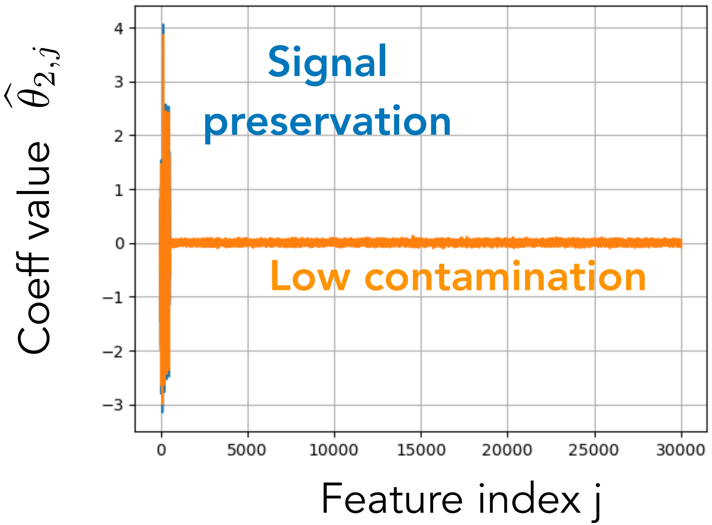


(k = 500, n = 5000, d = 30000, R = 100)



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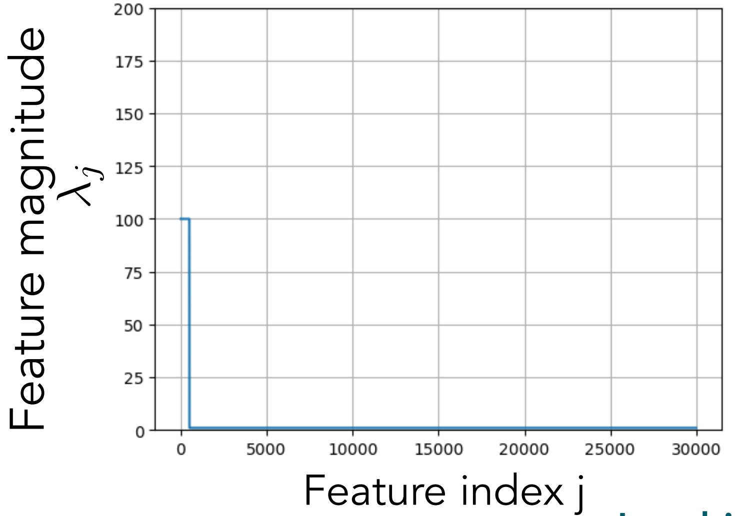
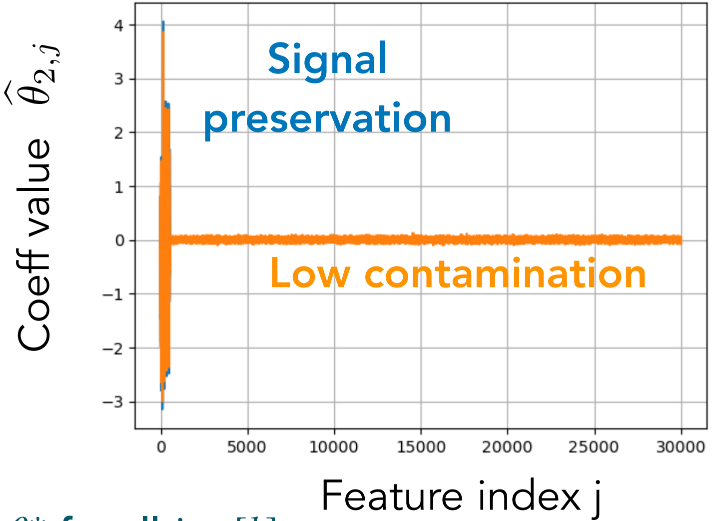


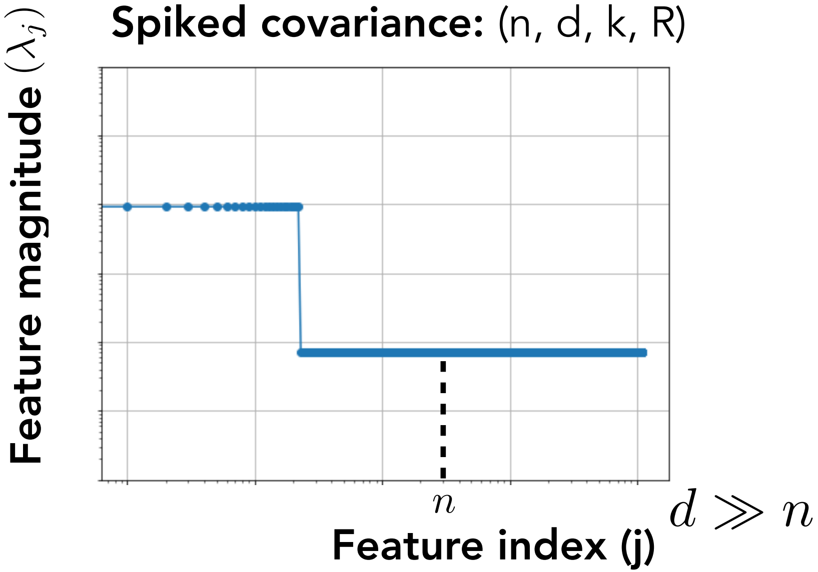


Low bias *iff* $\widehat{\theta}_j \approx \theta_j^*$ for all $j \in [k]$

Intuition: under near-orthogonality, $\widehat{\theta}_j \propto \sum_{i=1}^{n} y_i x_{i,j}$ - attenuation mitigated for larger R as $x_{i,j} \sim \mathcal{N}(0, R)$ for $j \in [k]$

A sensible model for l2: the **spiked-covariance** ensemble

$$\Sigma = \text{diag}(\Lambda) =$$



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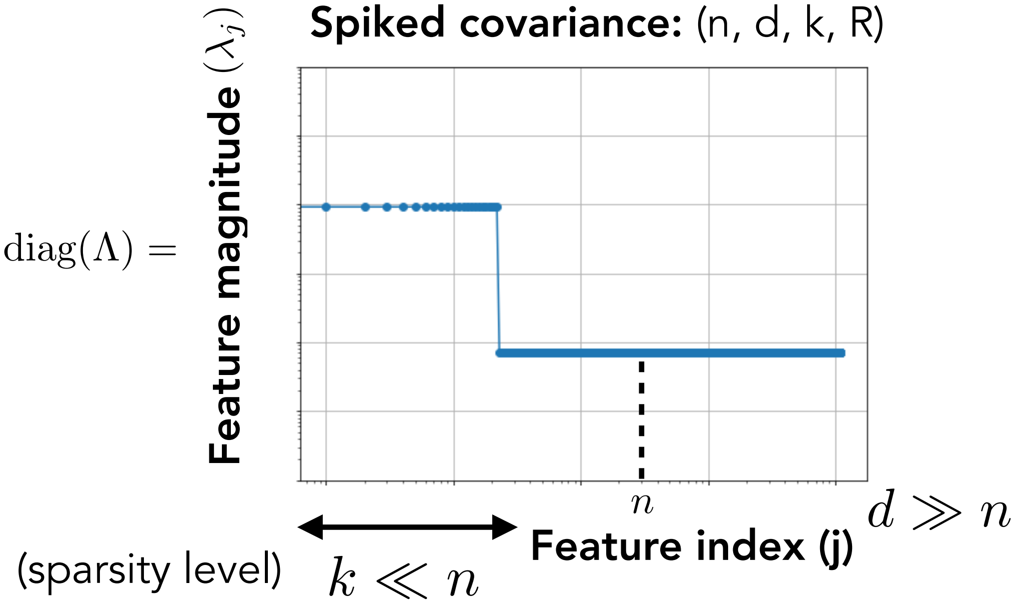

Conditions for **general anisotropic covariances** in terms of "effective ranks" by Bartlett et al (2020)

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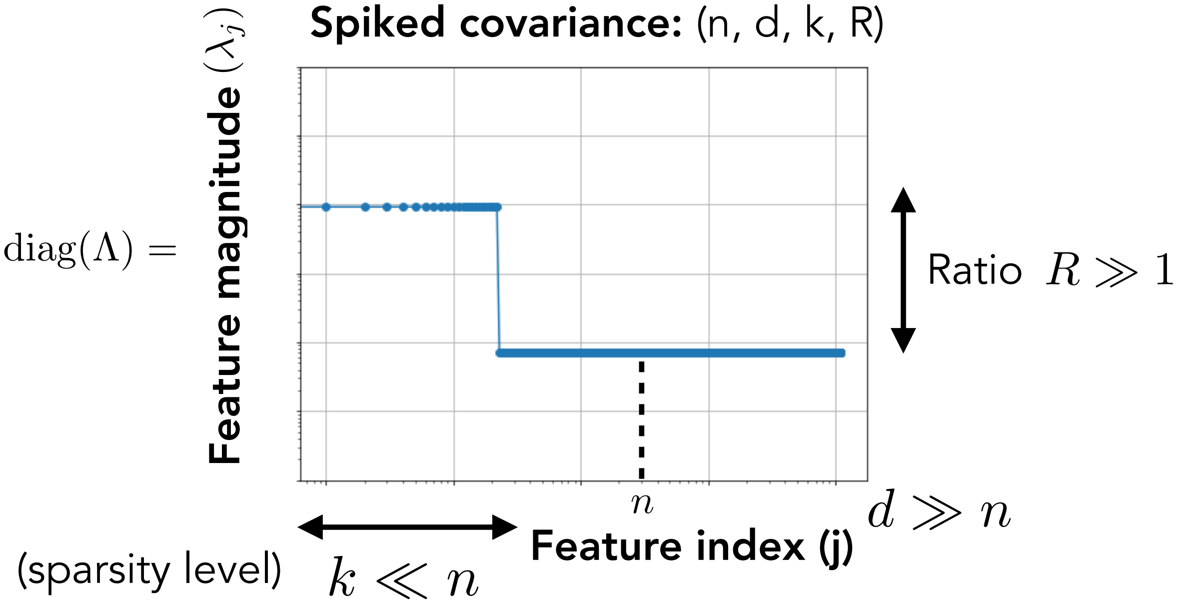

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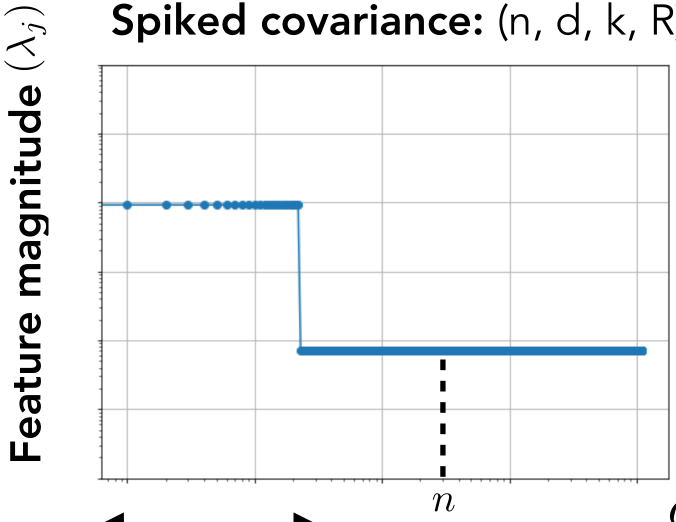

Ratio $R \gg 1$

$d \gg n$

(sparsity level) $k \ll n$

Additionally assume

$\theta_j^* = 0$ for all $j = k+1, \ldots, d$

A sensible model for l2: the **spiked-covariance** ensemble

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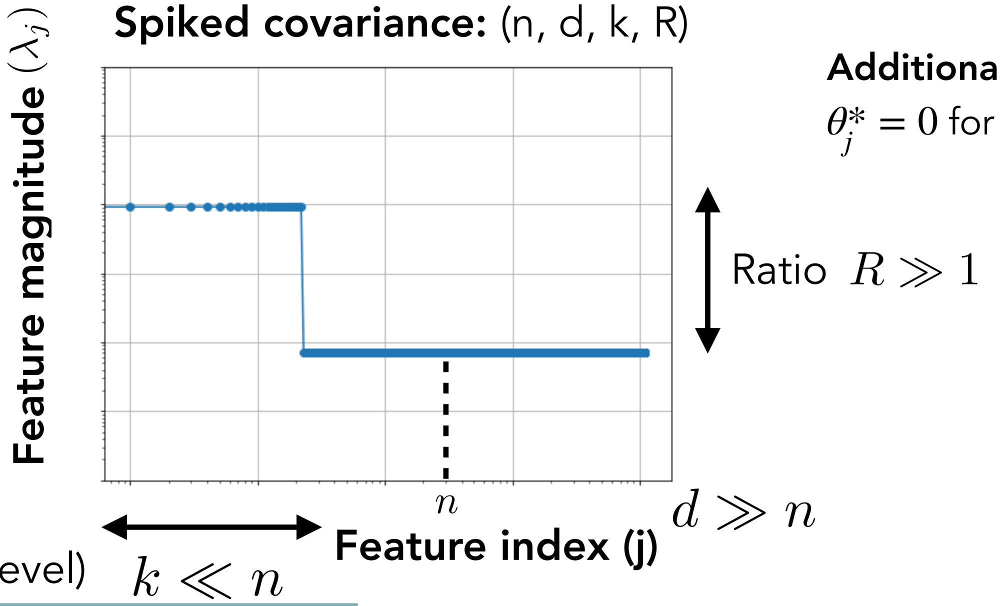

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Will **always** achieve

Variance $\to 0$ as $n, d \to \infty$:

Noise hidden along (d-k) directions!

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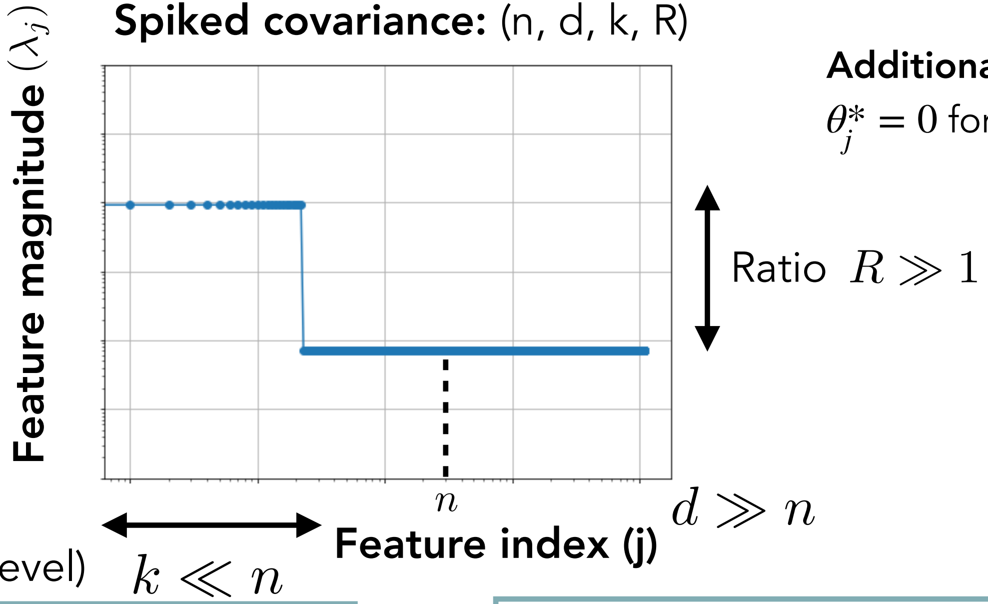

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Variance $\to 0$ as $n, d \to \infty$:
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Also achieves Bias $\to 0$ as $n, d \to \infty$
provided that $R \gg \frac{d}{n}$

Conditions for **general anisotropic covariances** in terms of "effective ranks" by Bartlett et al (2020)

Summary: Uniform benefits of overparameterization with spiked covariance

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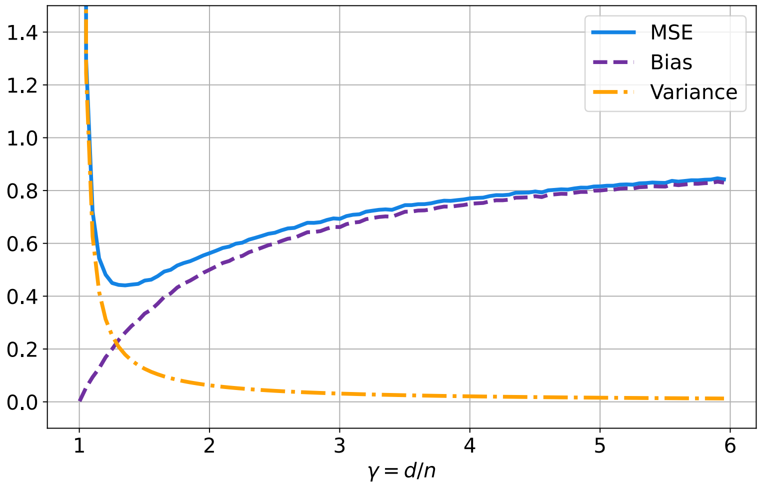
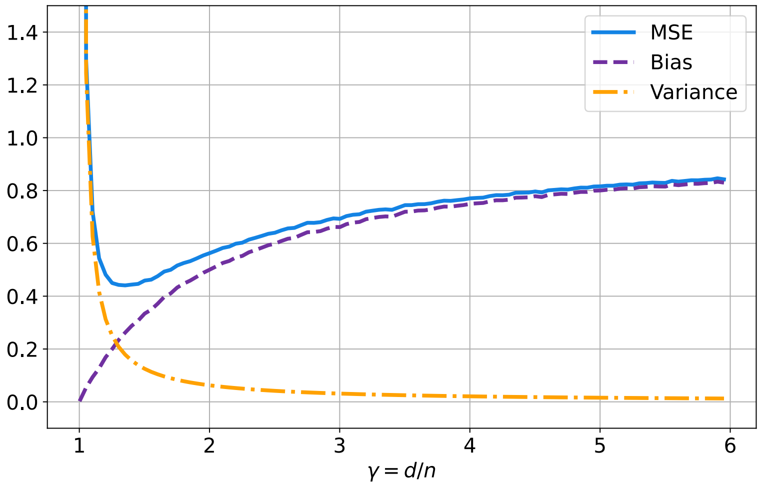


Isotropic covariance

Summary: Uniform benefits of overparameterization with spiked covariance

$$n = 500, \theta^* = \hat{e}_1, \sigma^2 = 0.25$$



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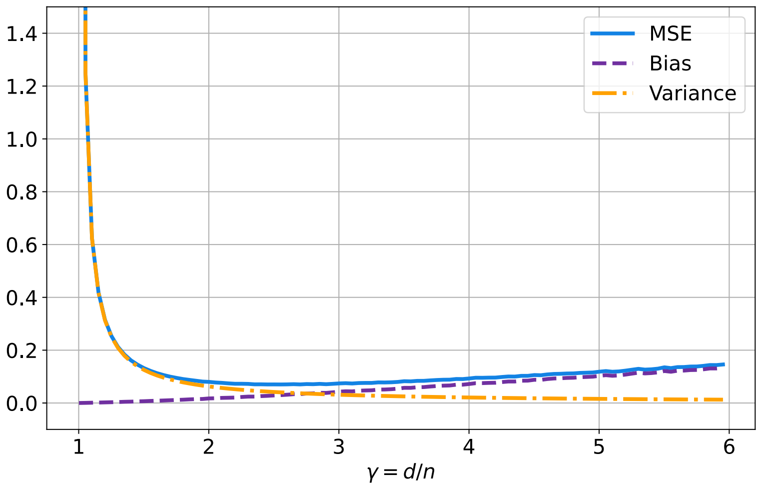

Spiked covariance, $R = 10$

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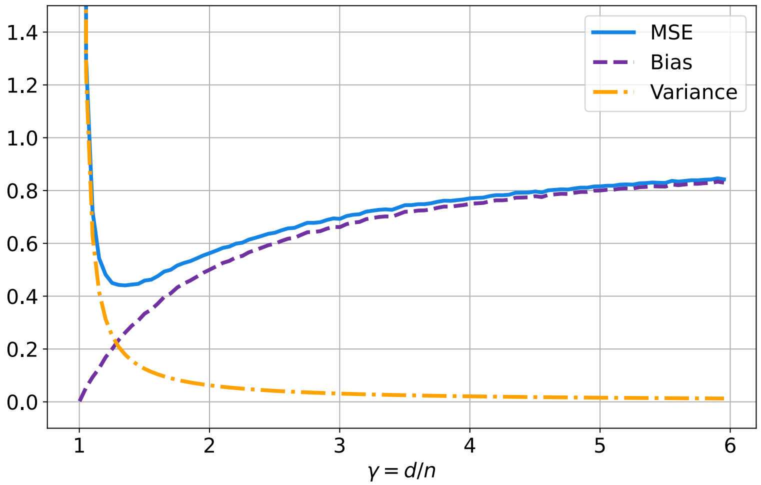

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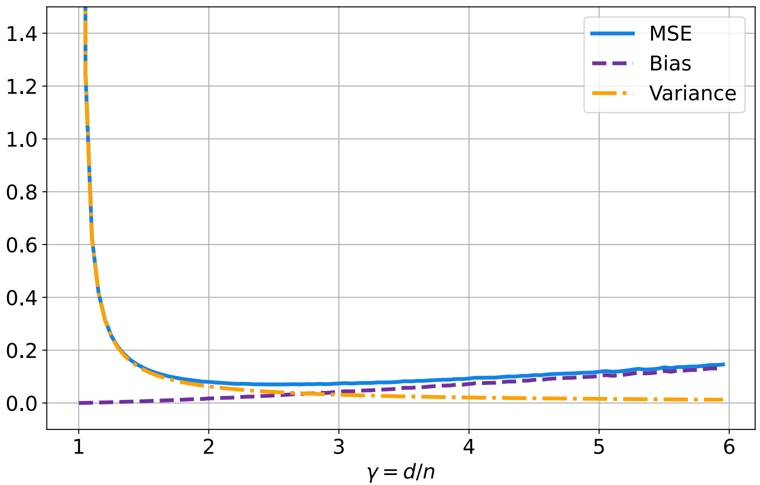
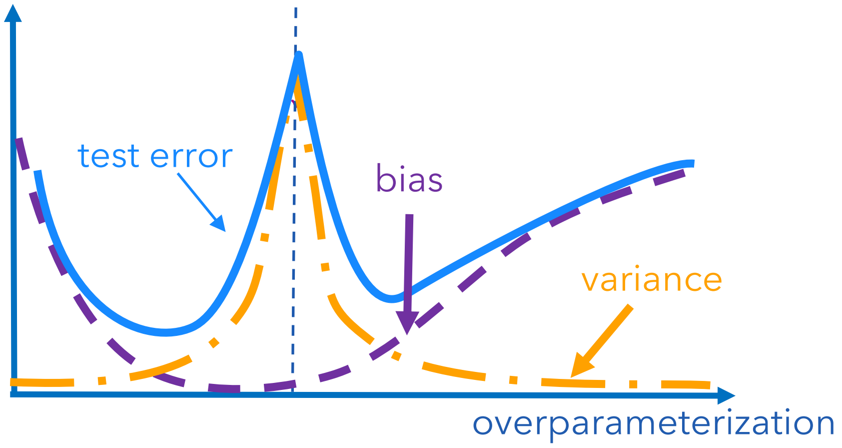


Spiked covariance, $R = 10$

For spiked covariance: (1) second descent ✓ (2) harmless interpolation ✓ (3) good generalization ✓

For fixed interpolator…



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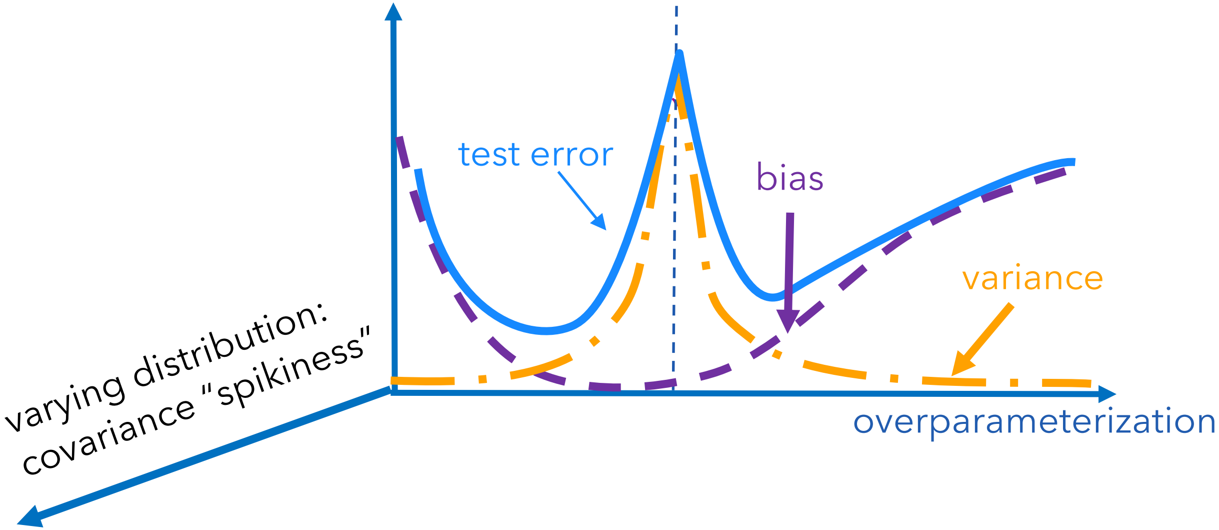

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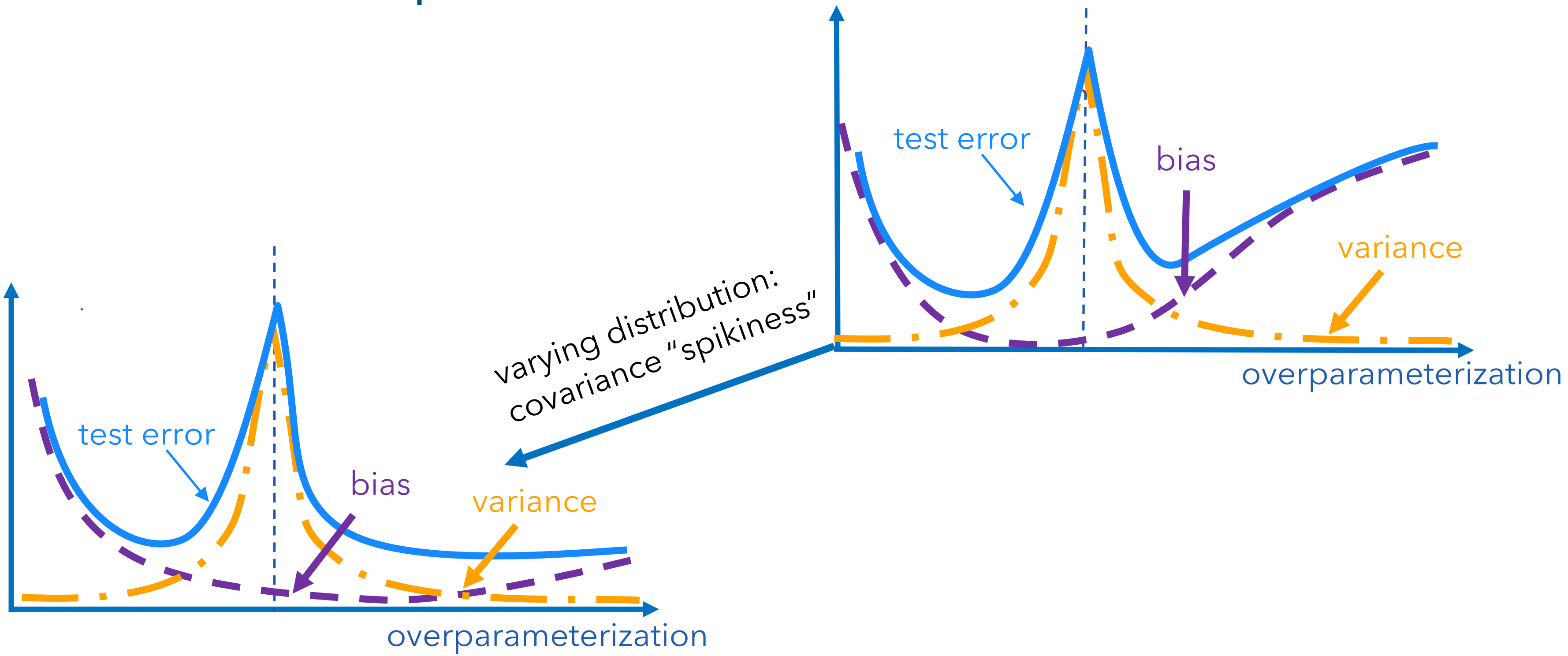

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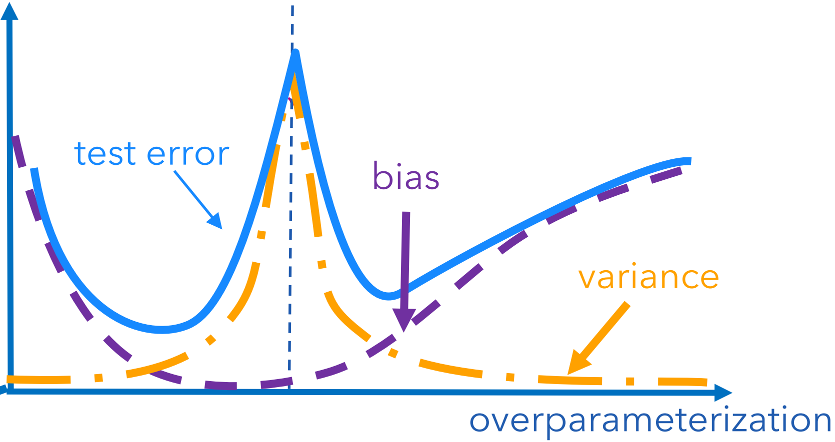

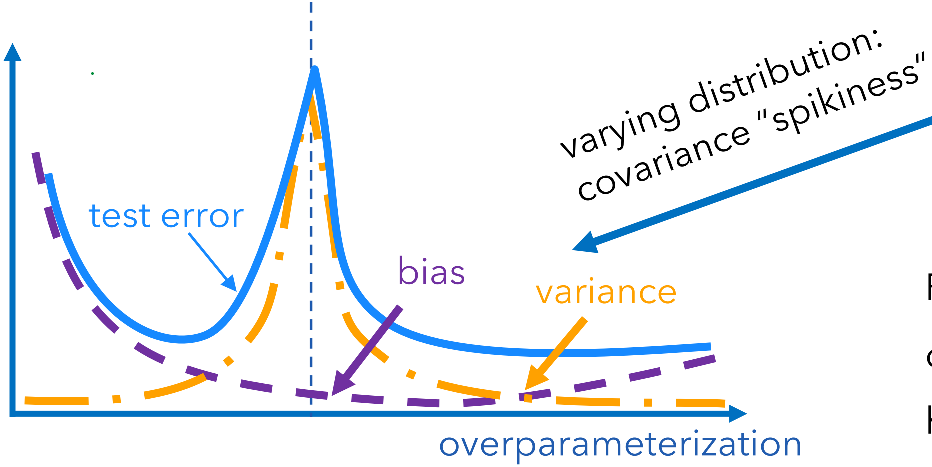

For a fixed distribution (e.g. isotropic),

different algorithms → different interpolators

how do bias and variance behave?

Plan today…

Part I: For linear regression, we discuss how

- variance can decay as overparameterization increases (simple math)
- Two factors can govern variance decay vs. bias increase
    - For fixed interpolator, certain problem instances/distributions are more benign
    - For fixed problem instance, certain interpolators generalize better

Part II: For classification, we discuss the

- effect of loss function choices
- implicit bias of optimization algorithms for neural networks
- generalization of neural networks on noisy, high-dimensional data

Implicit bias → inductive bias

opt. algorithm
minimizing loss

Implicit bias → inductive bias

opt. algorithm
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has
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e.g. 1st order method
on $||y - Xw||_2^2$

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e.g. for $p \in [1,2]$

$$\hat{w}_p = \text{argmin}_w \, ||w||_p$$
$$s.t. \, y = Xw$$

Implicit bias → inductive bias

opt. algorithm minimizing loss —has implicit bias towards→ certain interpolator —has certain strength of inductive bias towards→ certain structure

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Next: Recall how as $p \to 1$ has an inductive bias towards sparse solutions

Recall: Inductive bias for sparse linear models

isotropic

Fixed distribution: $y_i = \langle w^\star, x_i \rangle + \xi_i$ with **sparse** $w^\star$, i.e. $||w||_0 = k \ll d$, i.i.d. noise ξ_i and $x_i \sim N(0, I)$

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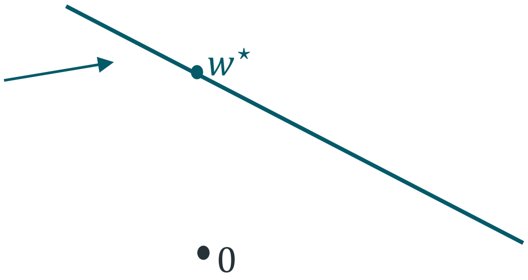
subspace of all linear interpolators $\{w \colon Xw = y = Xw^\star\}$

for noiseless $\xi_i = 0$



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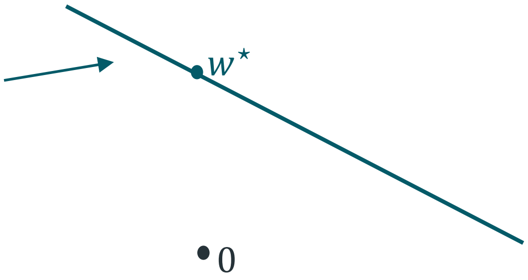

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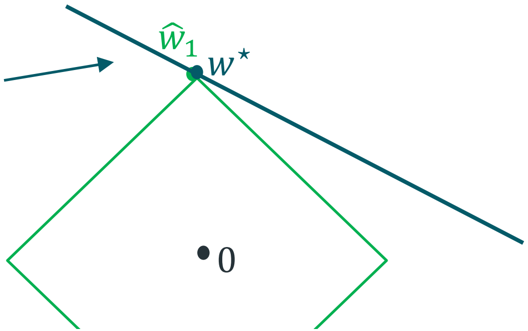

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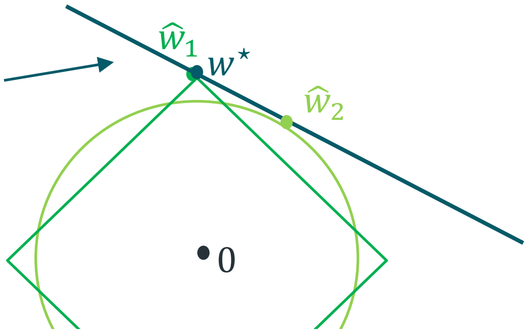

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$y = Xw^\star$

Basis pursuit: $\hat{w}_1 = \text{argmin}_w \, ||w||_1 \; s.t. \; y = Xw$

Perfect recovery
w.h.p. for $n \sim k \log d$

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when observations are noisy

Noisy
$y = Xw^\star + \xi$

Lasso: $\hat{w}_\lambda = \text{argmin}_\text{w} \, ||y - Xw||_2^2 + \lambda ||w||_1$

Estimation error achieves
minimax optimal rate
$O\left(\frac{k \log d}{n}\right)$ for best λ

e.g. BP: [Candes, Tao '05, Donoho '06], Lasso: [Bunea, Tsybakov, Wegkamp '07, vandeGeer '08], [Wainwright '09]

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	↓ when observations are noisy	
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➡ $p = 1$ has a strong inductive bias towards sparse solutions → small *statistical* bias!

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Previously unknown: prediction/estimation error of min-ℓ_1 interpolation for **noisy data**

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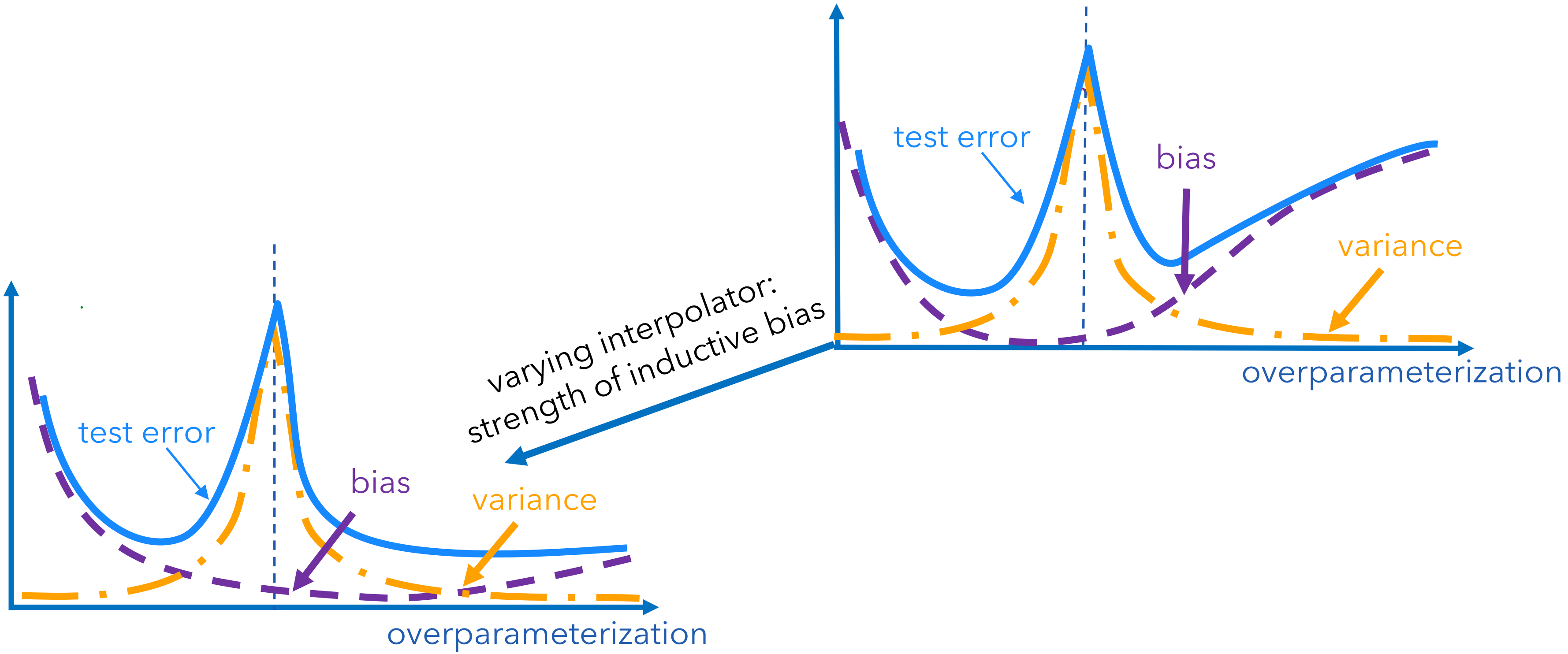
For fixed distribution…



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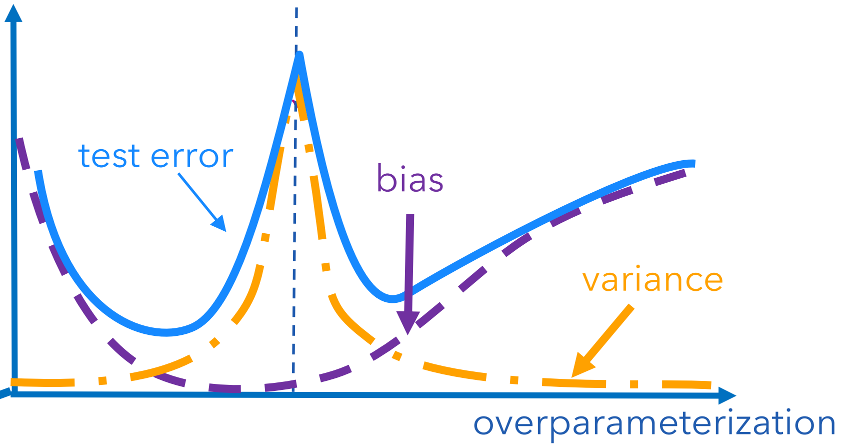

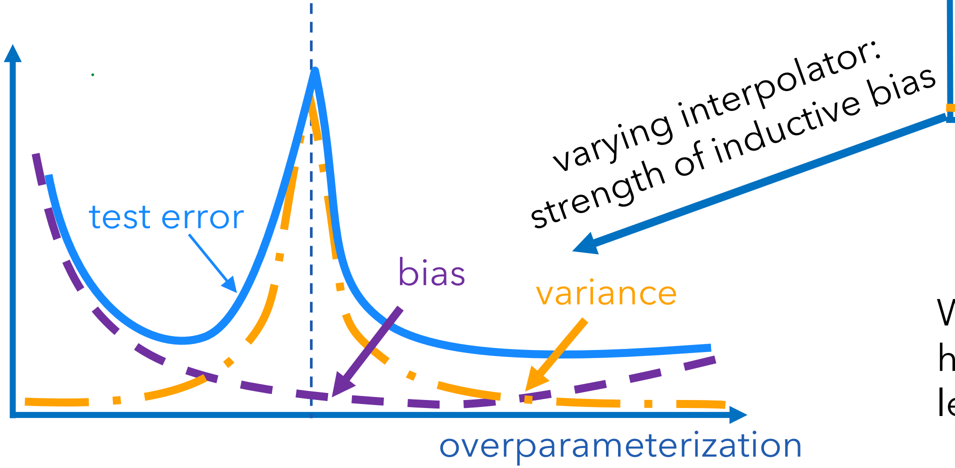

When interpolating noise,
how strong of an inductive bias
leads to good generalization

Inductive bias for noisy sparse linear models

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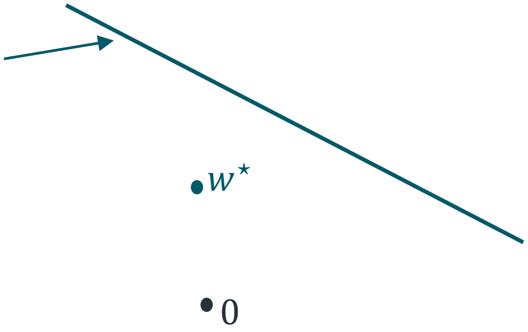

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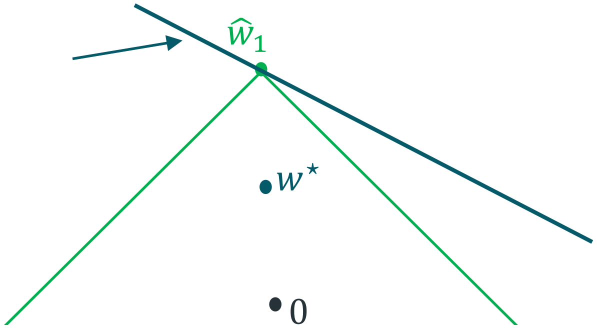

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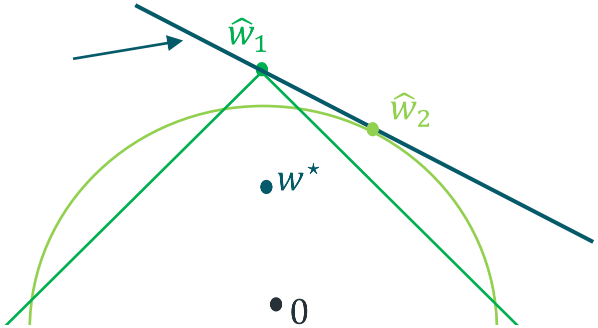

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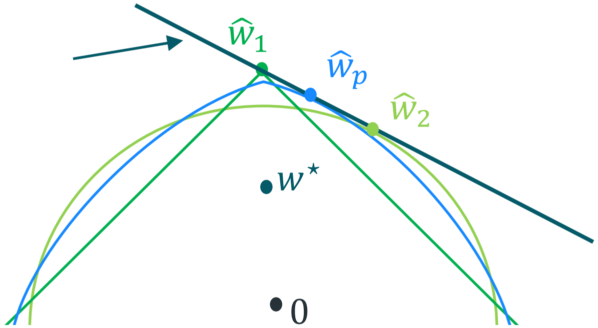

Varying inductive bias via $p \in [1,2]$

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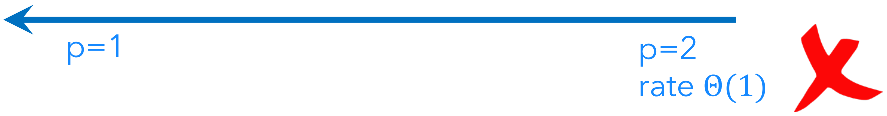
strong inductive bias towards sparsity ← p=1 ———— p=2 no inductive bias towards sparsity

Strong inductive bias: $p = 1$



Inconsistent

but harmless interpolation

Strong inductive bias: $p = 1$

decreasing statistical bias ←

p=1

p=2
rate $\Theta(1)$

✗ Inconsistent

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- Tight bounds for adversarial noise vectors ξ but $O(\sigma^2)$ for ξ_i i.i.d. with variance σ^2
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Consistent

but harmful interpolation:

opt. regularized $O\left(\frac{k \log n}{n}\right)$

decreasing statistical bias

p=1

rate $\Theta\left(\frac{1}{\log n}\right) = \tilde{\Theta}(1)$

p=2

rate $\Theta(1)$

Inconsistent

but harmless interpolation

The problem of $p = 1$ lies in the variance…

For $p = 1$ and $k = 1$, "sensitivity to noise" and variance larger than for $p = 2$

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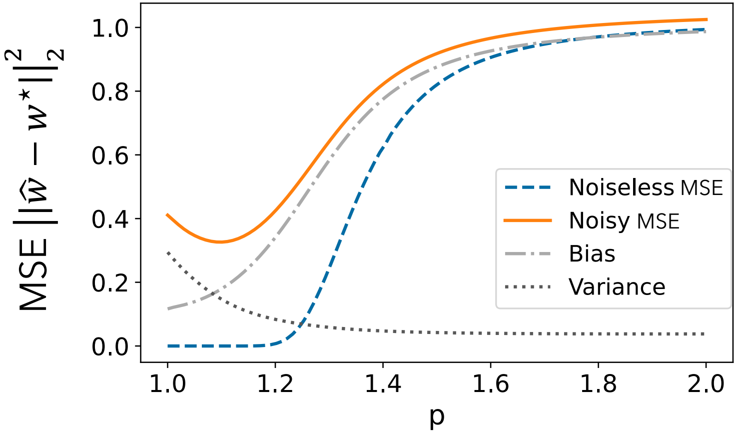

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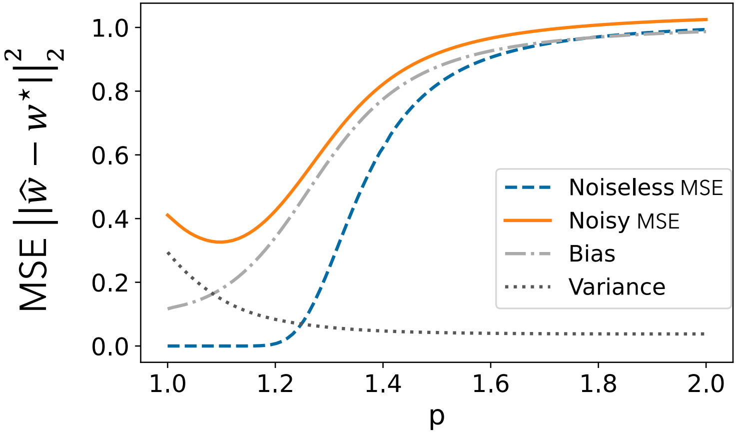

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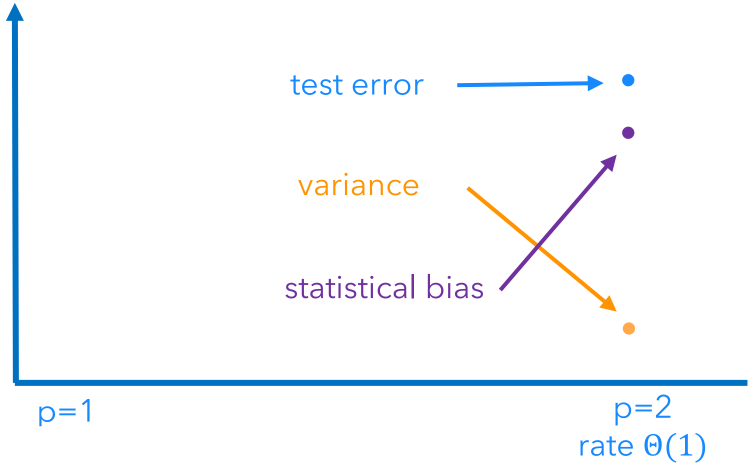
→ as overparameterization increases, variance decay is slower for $p = 1$ than for $p = 2$!

A bias-variance trade-off for $p \in [1,2]$

Min-ℓ_p-norm interpolation $\hat{w}_p = \text{argmin}_w \, ||w||_p \;\; s.t. \, y = Xw$



strong inductive bias towards sparsity

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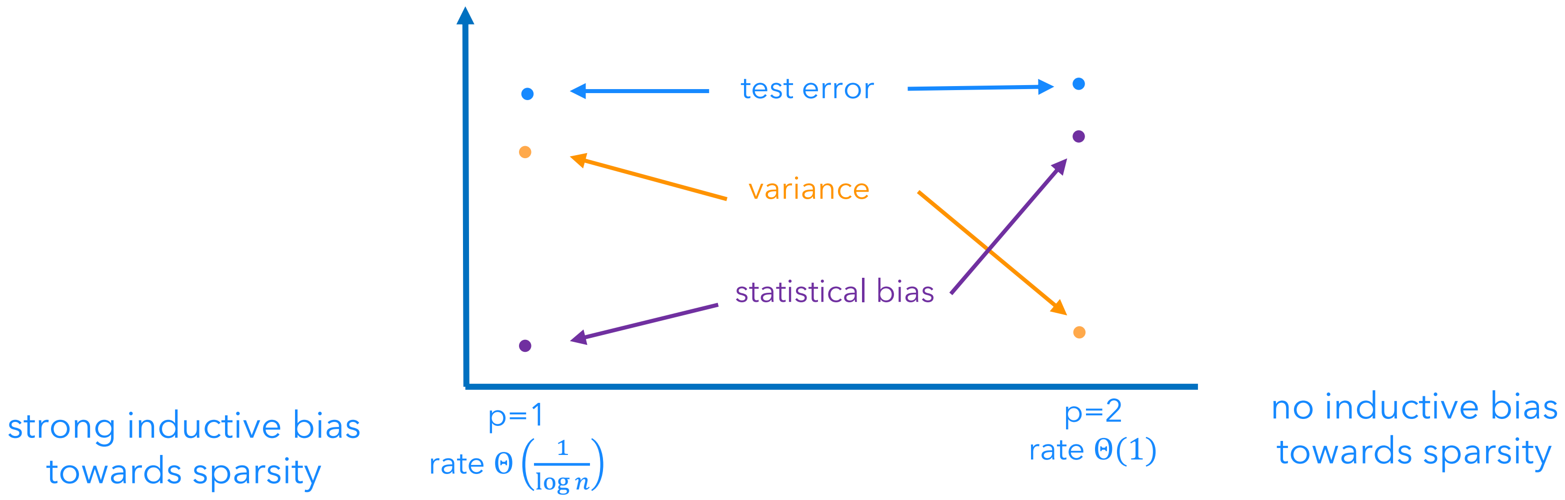

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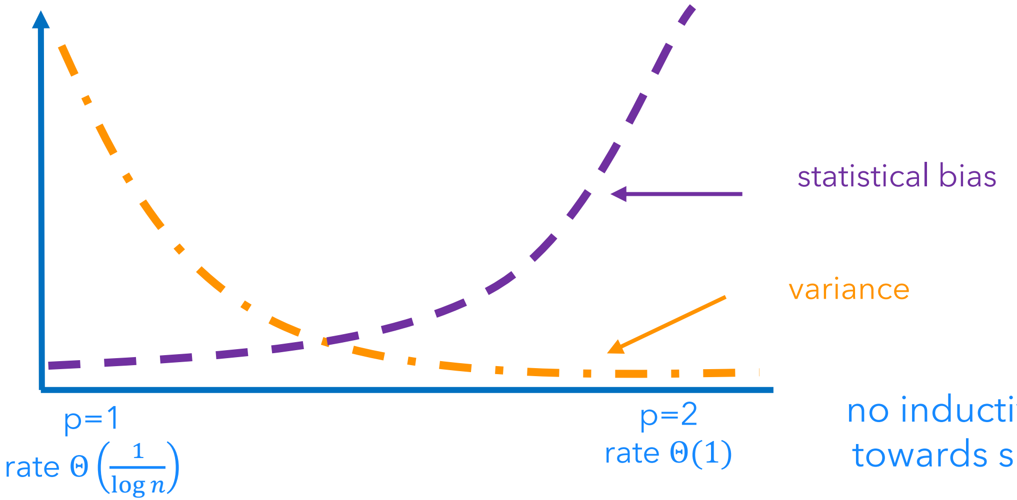

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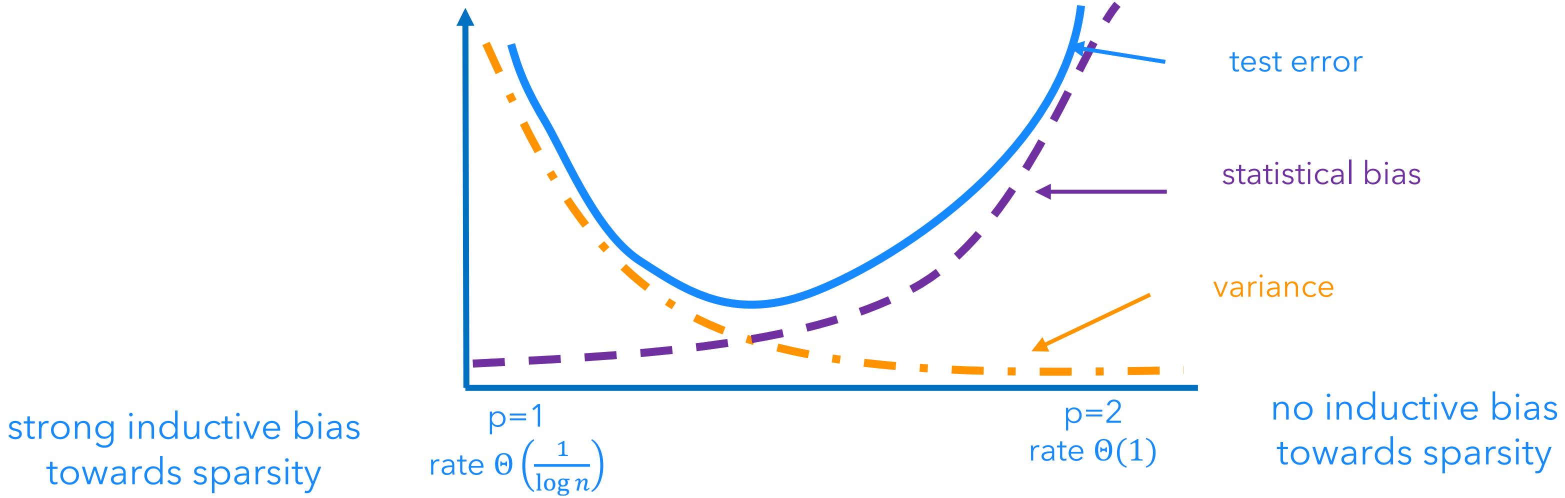

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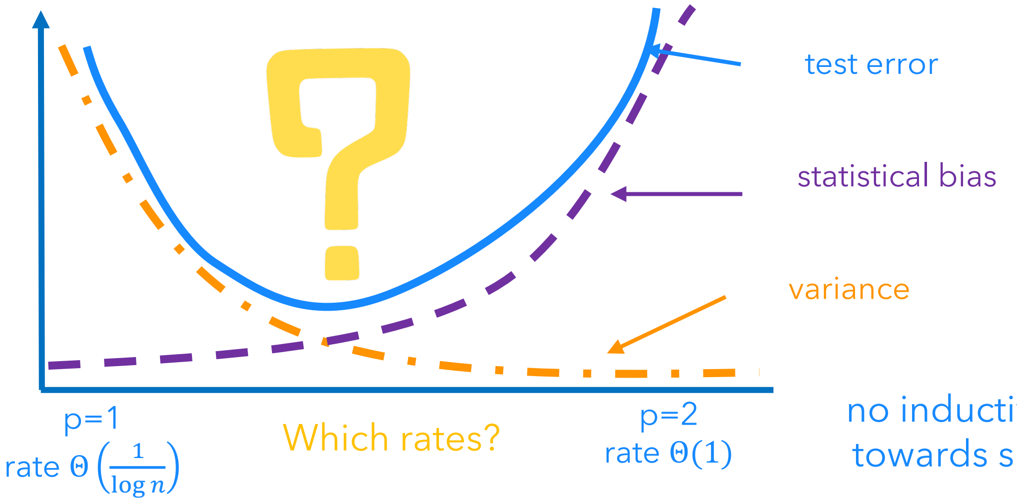

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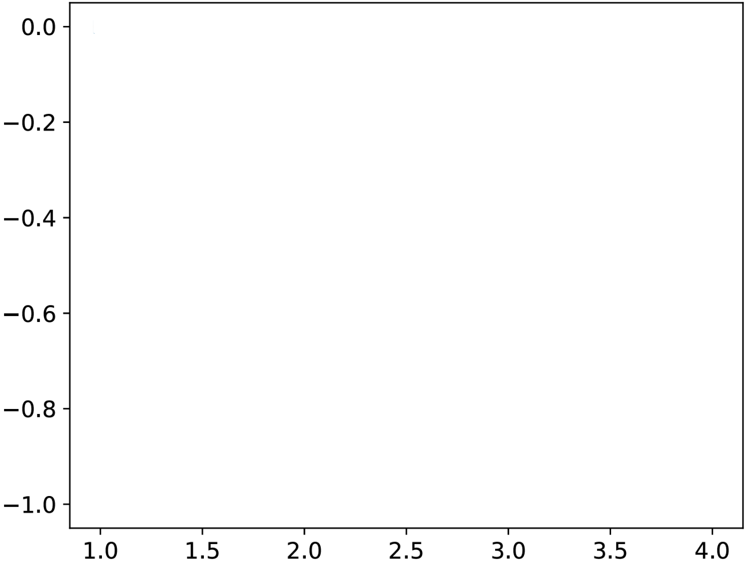
Tight bounds for $p \in [1, 2]$



degree of overparameterization β: $d \asymp n^{\beta}$

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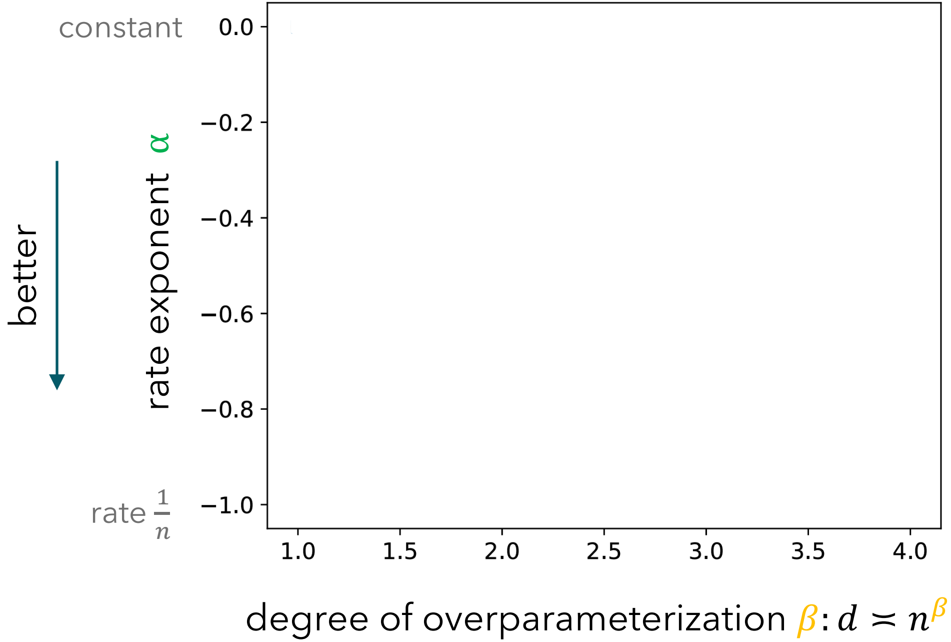

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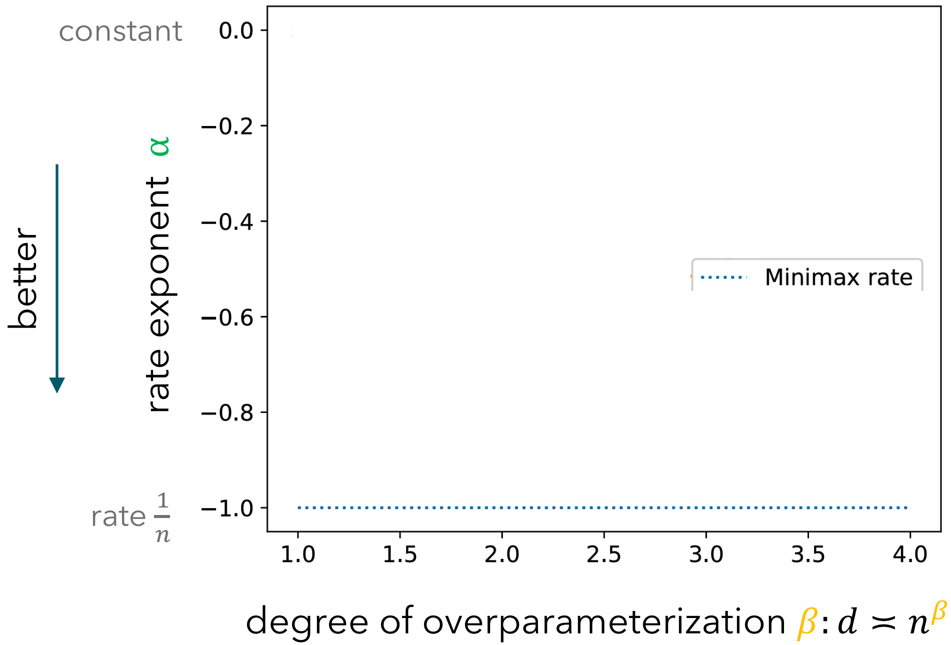

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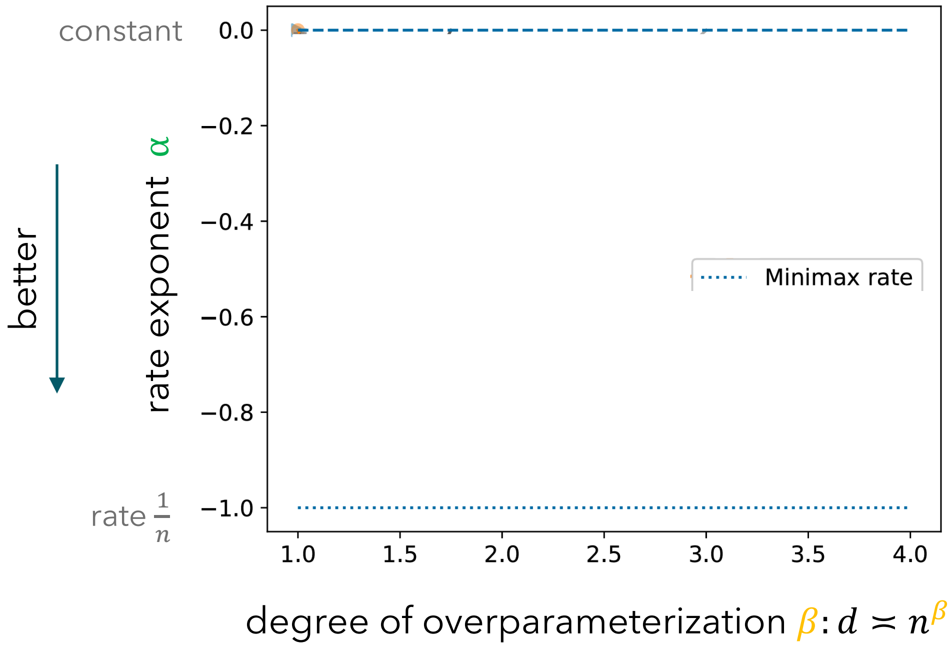

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For $p \geq 1$: [Wang, Donhauser, Yang '22]

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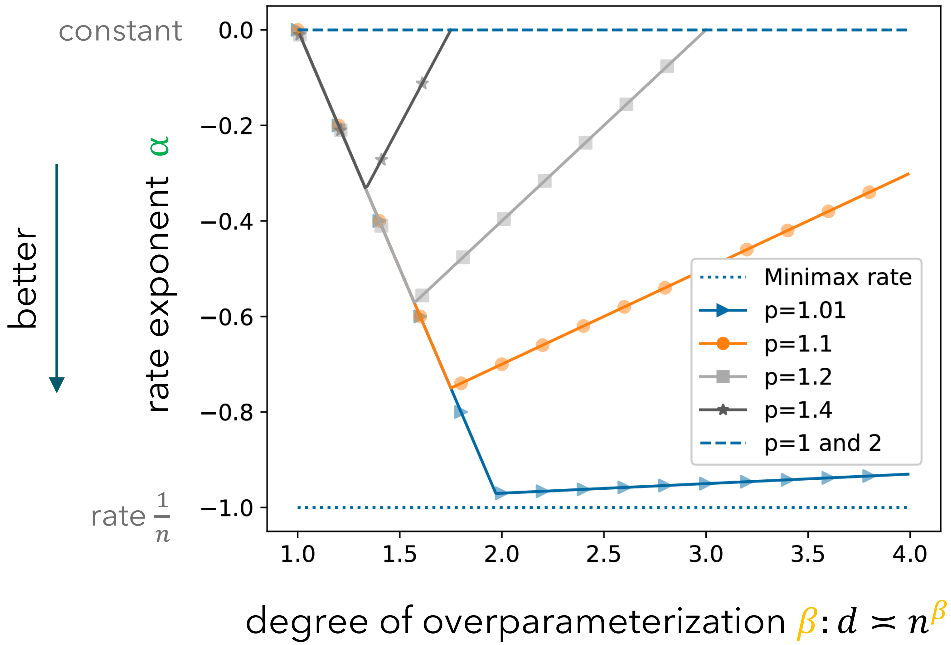

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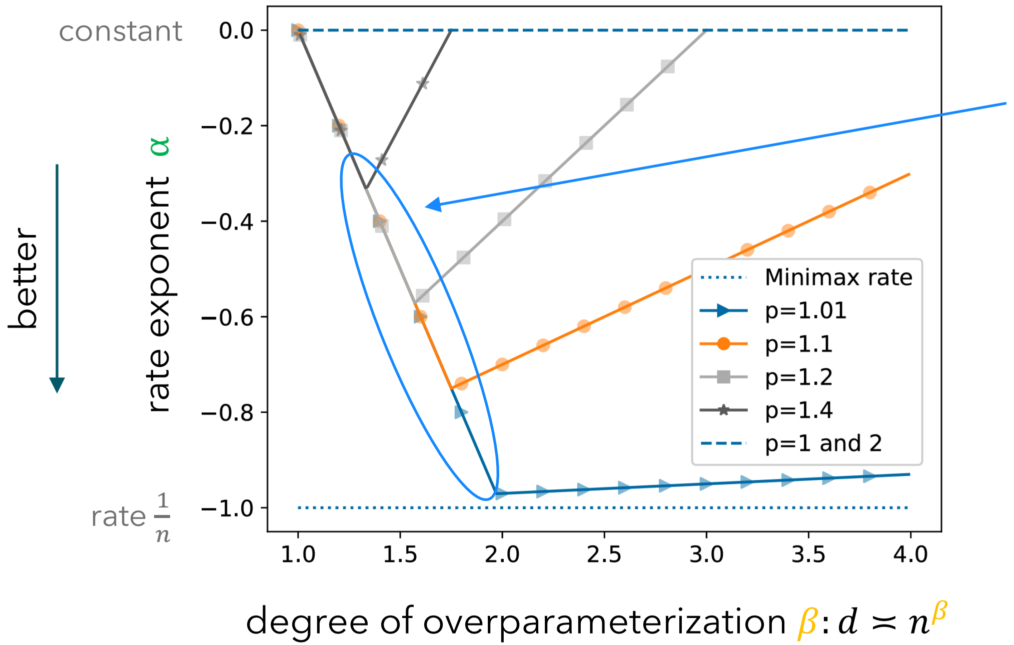

"second" descent:
decrease due to
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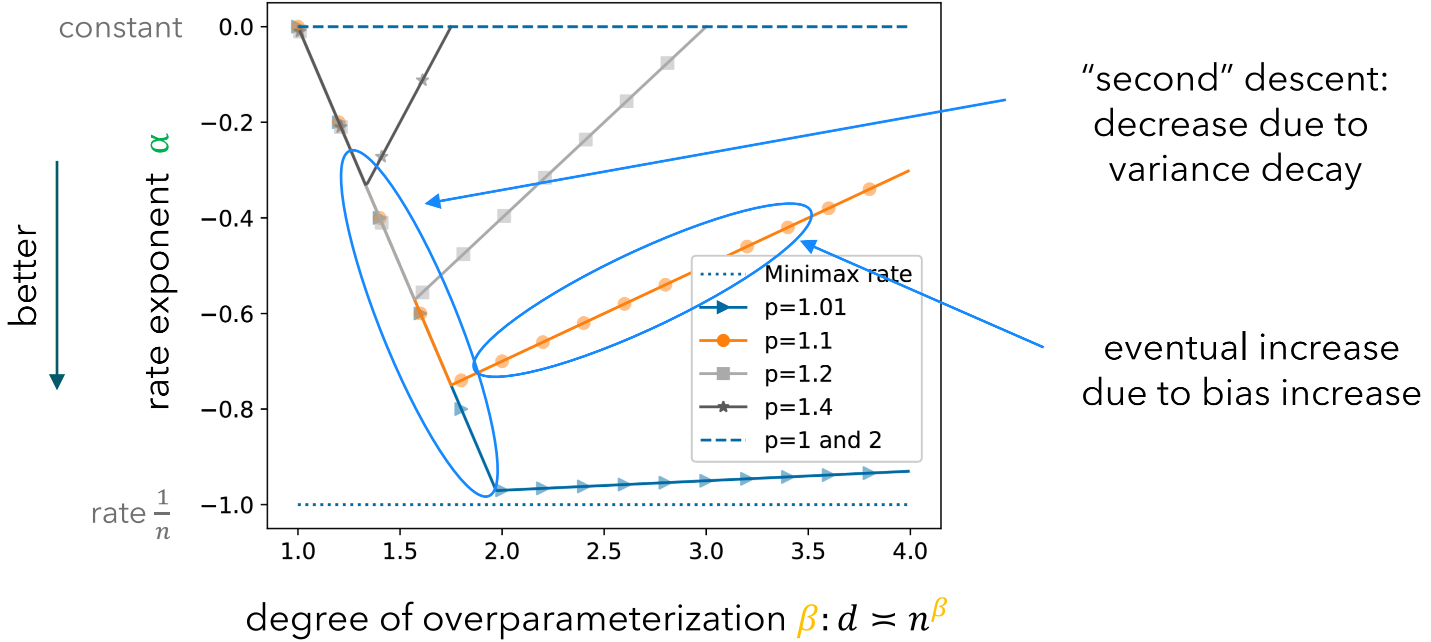

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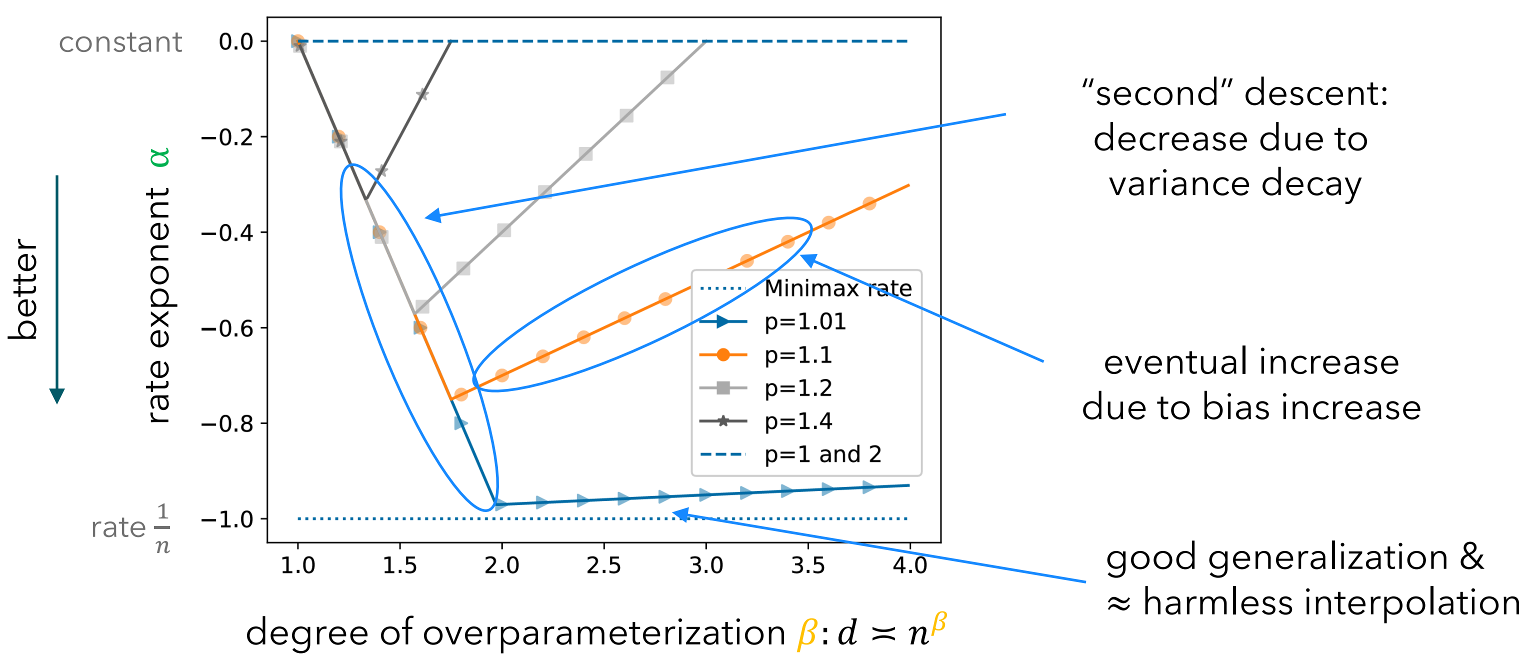

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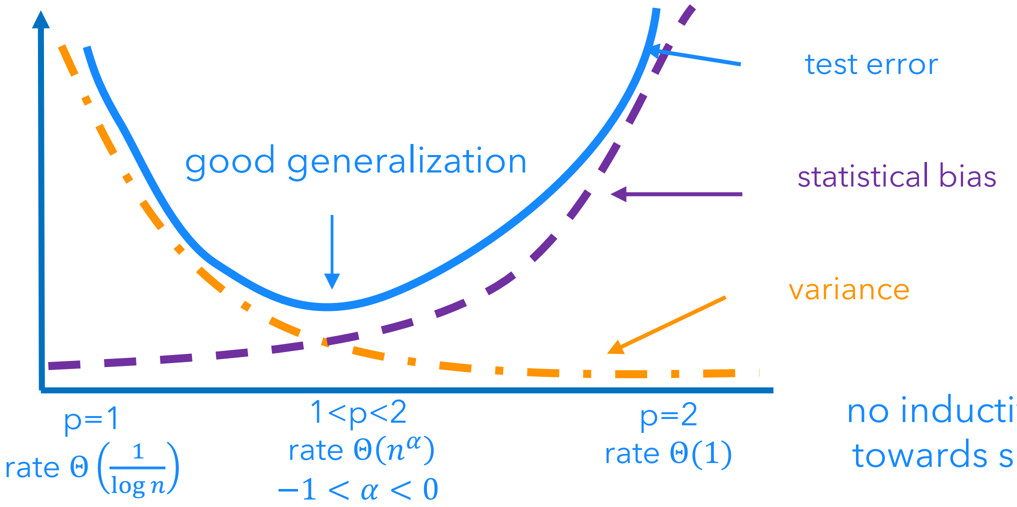

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Take-away: medium strength of inductive bias is best when interpolating noise

How transferable is this "new" intuition?

- Proof technique using Convex Gaussian Minmax Theorem [Thrampoulidis, Oymak, Hassibi '15] with localized convergence [Koehler, Zhou, Sutherland, Srebro '21] carries over to lin. classification

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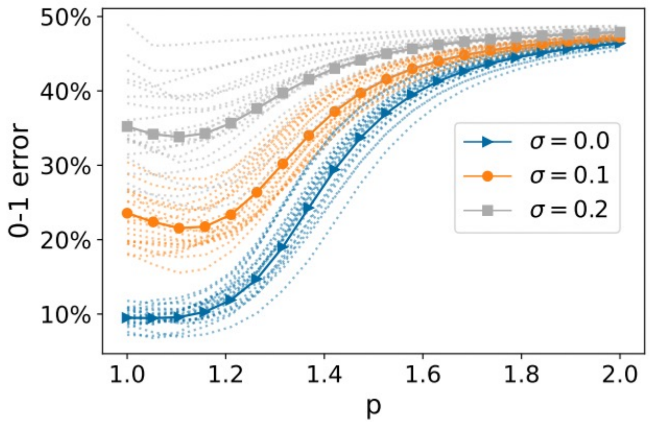

Synthetic experiment:
Isotropic Gaussians with $d \sim 5000, n \sim 100$

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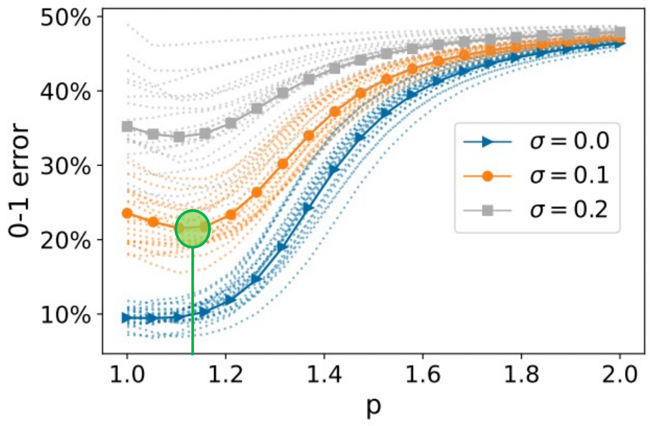

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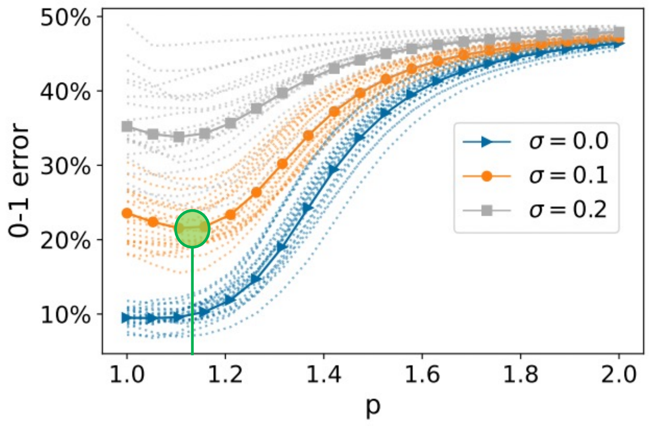

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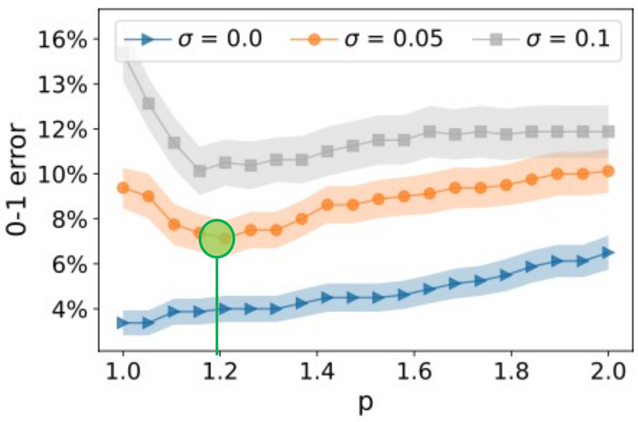

Real-world experiment:
Leukemia dataset with $d \sim 7000, n \sim 70$

[Stojanovic, Donhauser, Yang '22], [Donhauser, Ruggeri, Stojanovic, Yang '22]

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Nonlinear structure: Rotational invariance for WideResNet

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- strength of rotational invariance via "amount of" data augmentation

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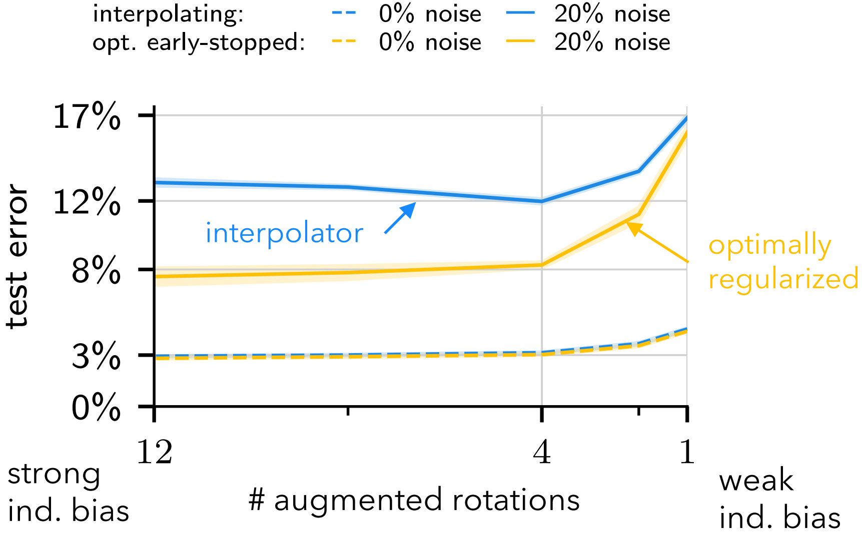

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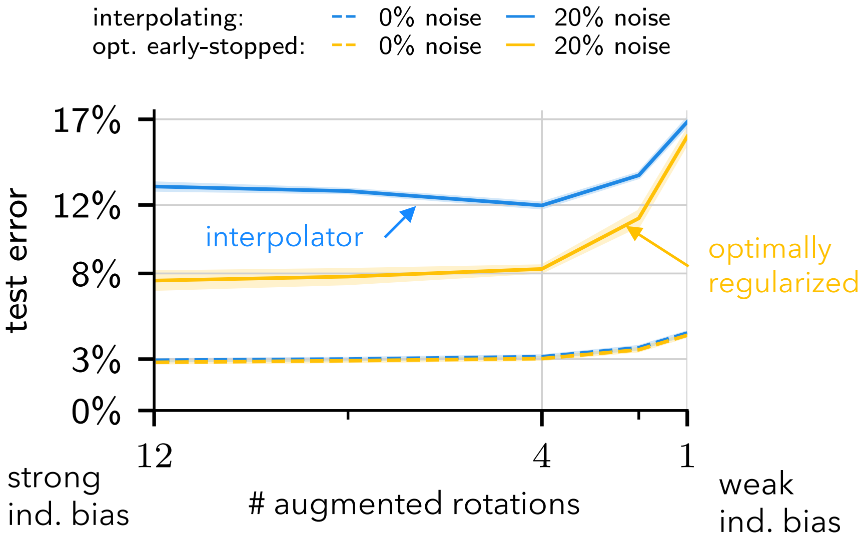

Confirmed: medium strength of inductive bias is best when interpolating noise

[Aerni, Milanta, Donhauser, Yang '23]

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  open: comprehensive experimental NN study!

Plan today…

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Classification-vs-regression: A tale of two loss functions

	0-1 loss	Squared loss
Logistic loss		
Squared loss		Regression

Classification-vs-regression: A tale of two loss functions

	0-1 loss	Squared loss
Logistic loss	Classification, most popular	
Squared loss	Classification, less popular	Regression

Differences in training loss functions

Squared loss

↓ **Gradient descent** (Folklore, see e.g. Engl et al 1996)

Minimum-l2-norm interpolation

$$\widehat{\theta}_2 = \arg\min \|\theta\|_2$$
$$\text{subject to}$$
$$X_i^\top \theta = Y_i, i \in [n].$$

- Closed-form
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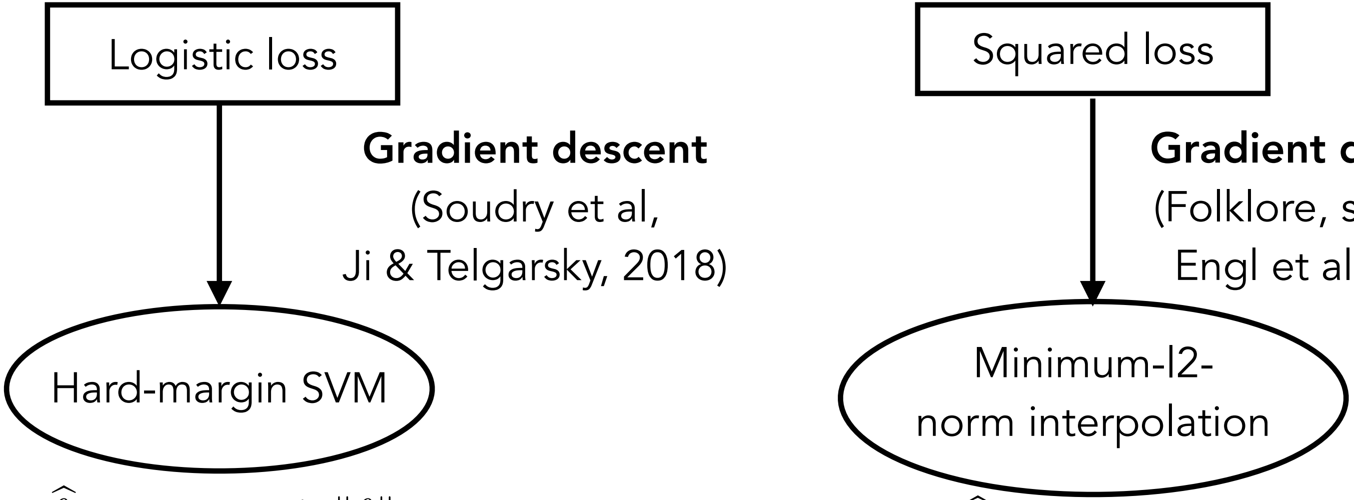

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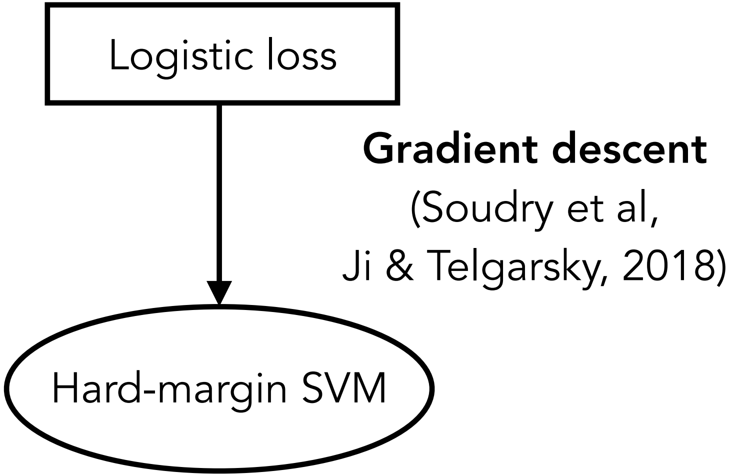

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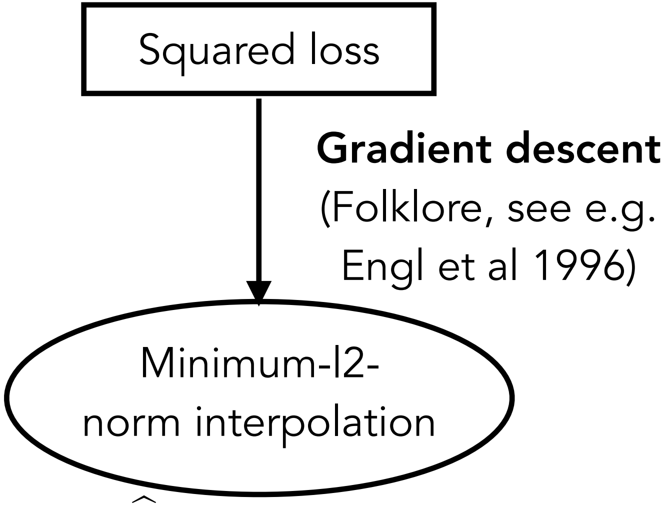

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Differences in test loss functions

Regression: Test MSE

$$\mathcal{E}_{\mathsf{MSE}} = \mathbb{E}\left[(X^\top(\widehat{\theta} - \theta^*))^2\right]$$

Classification: Test 0-1 error

$$\mathcal{E}_{0-1} = \mathbb{E}\left[\mathbb{I}[\mathrm{sgn}(X^\top\widehat{\theta}) \neq \mathrm{sgn}(X^\top\theta^*)]\right]$$

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Two core challenges when analyzing classification:

1. Hard-margin SVM does not have a closed-form solution, unlike minimum-l2-norm interpolation
2. 0-1 error metric challenging to sharply analyze as compared to MSE

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Implication: SVM has a closed-form expression, can be more easily analyzed!

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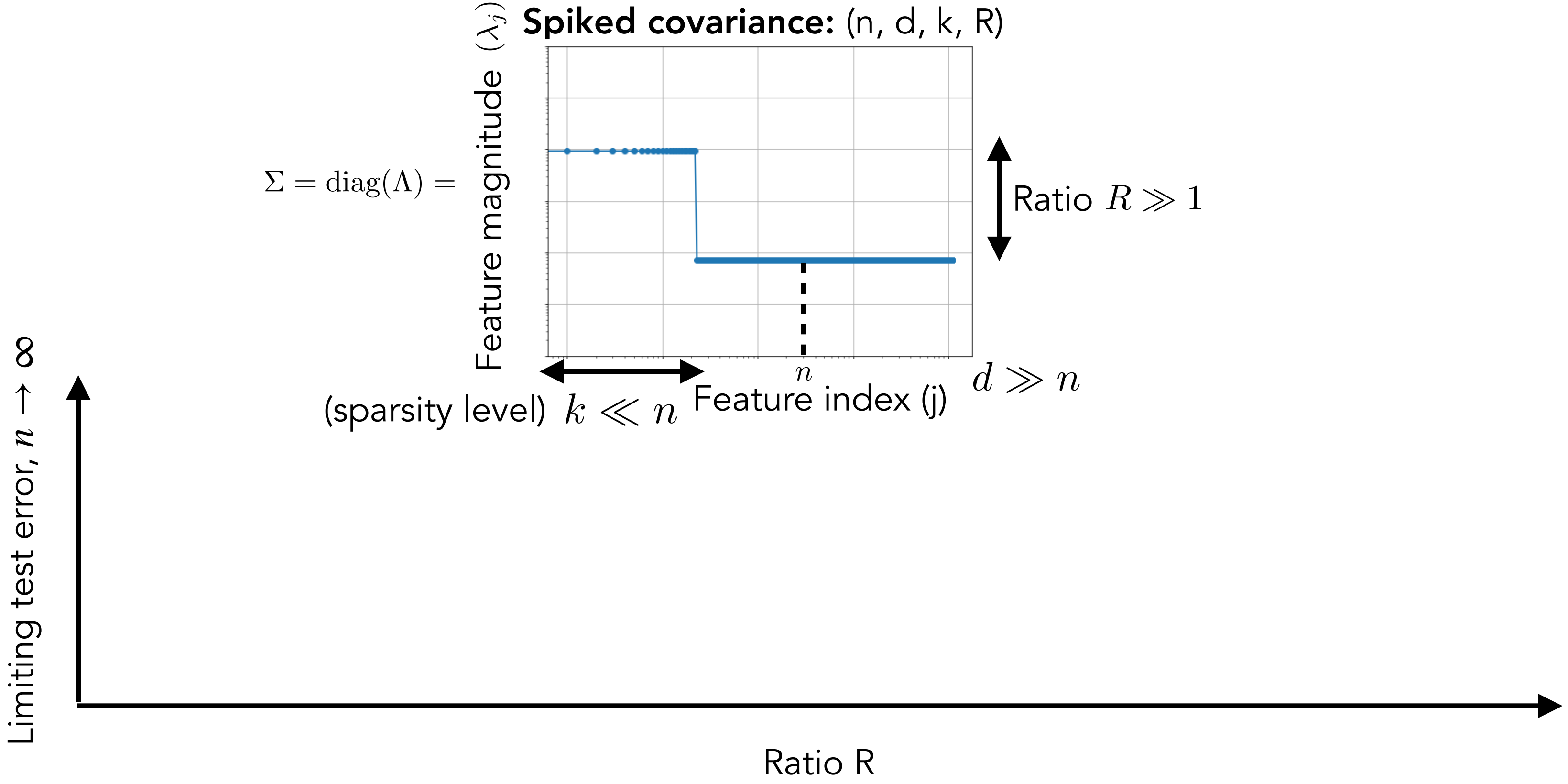
One analysis path for l2, step 2: analyzing 0-1 error of **interpolator**



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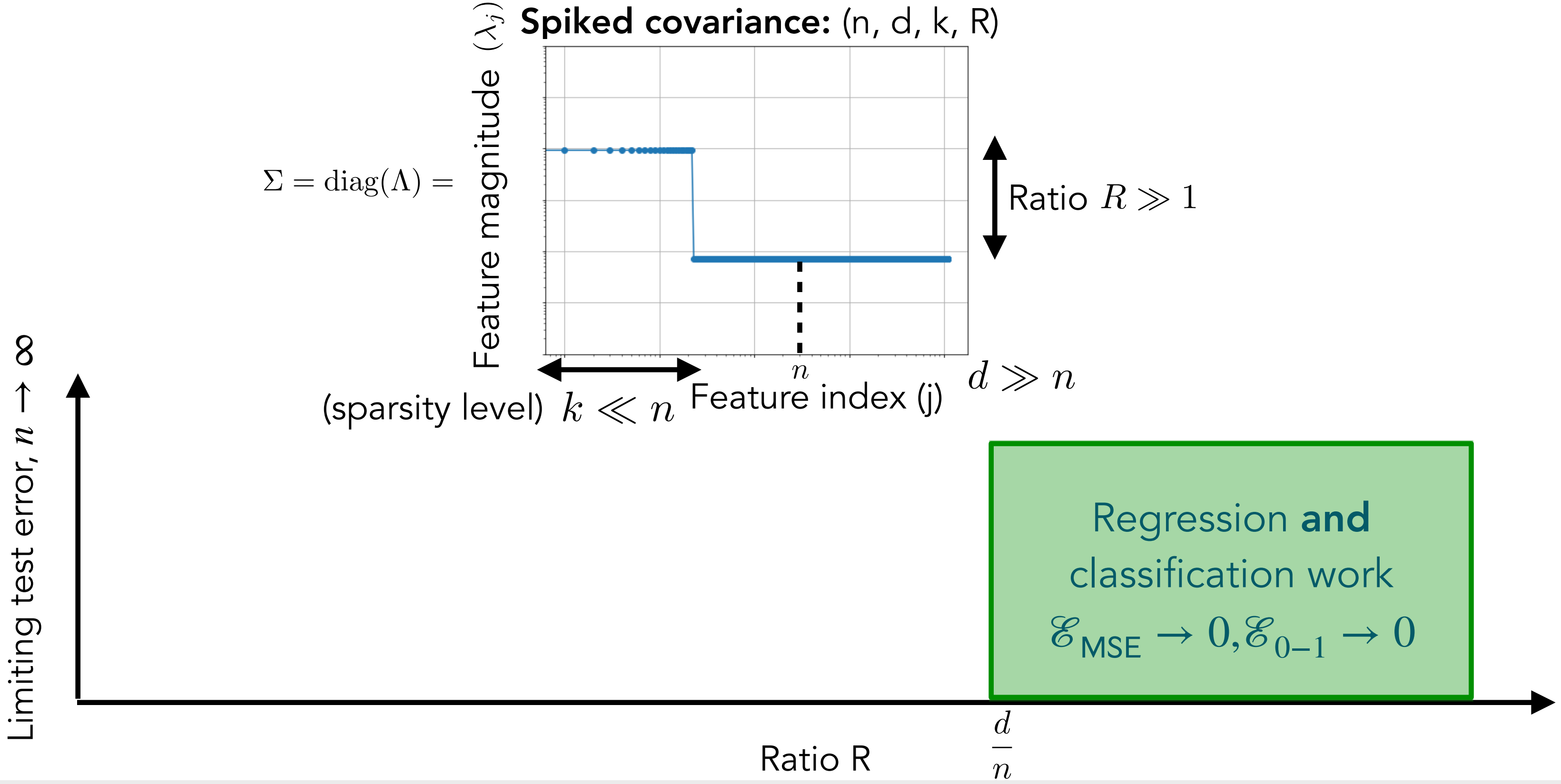

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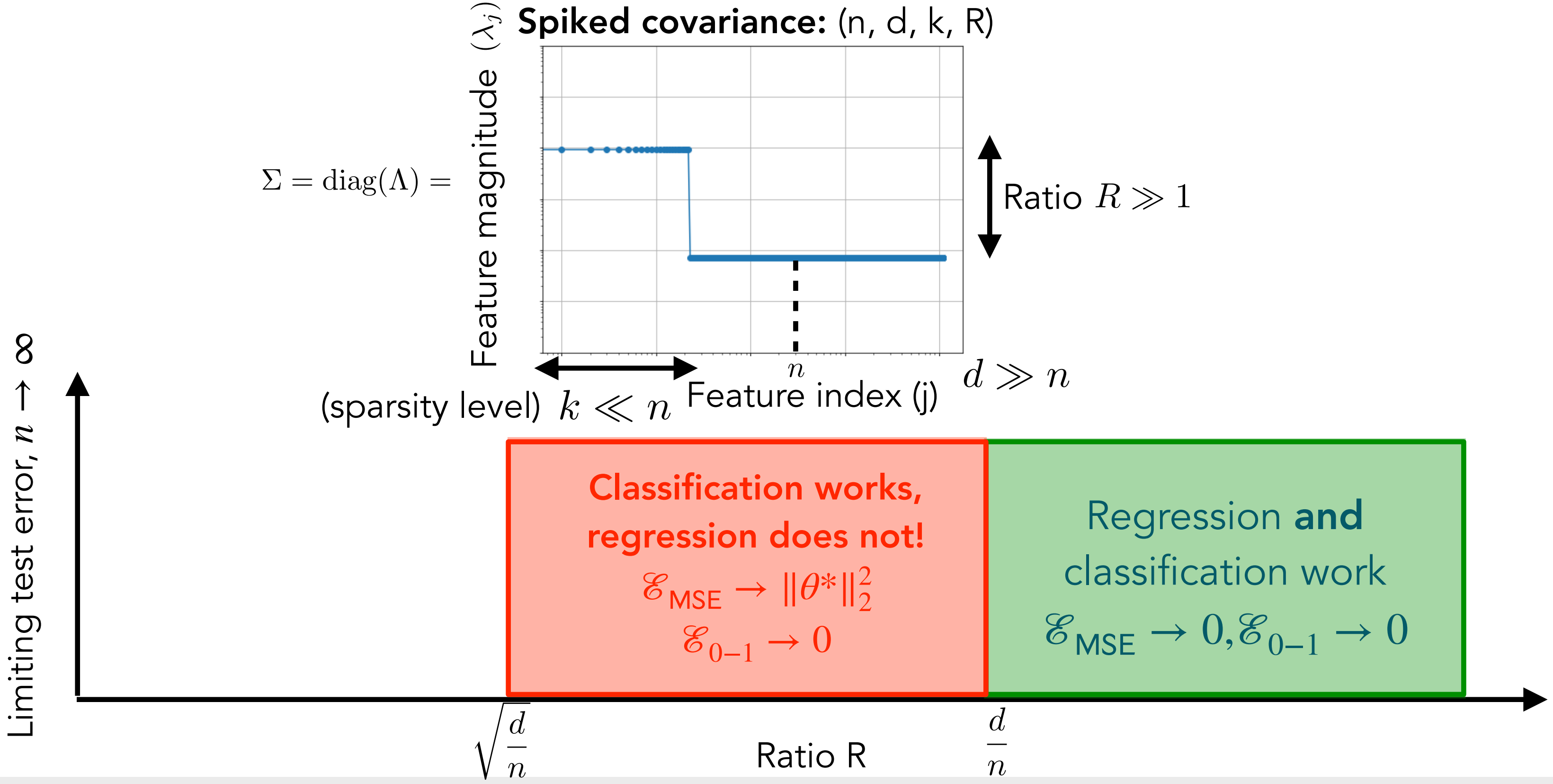

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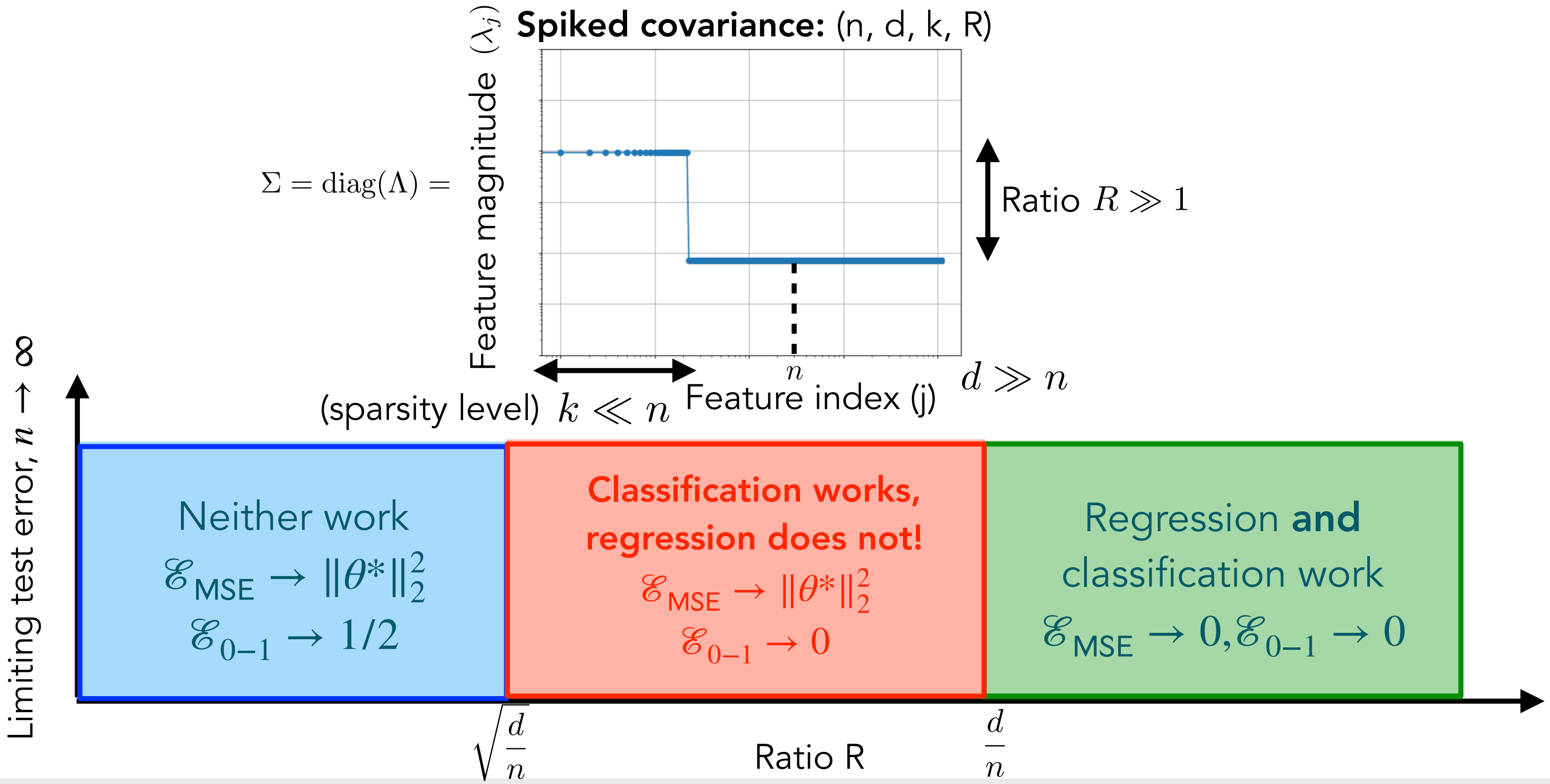

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Takeaways for classification with l2-minimizing solutions

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- Different **training loss functions** could yield **similar or even identical solutions**
- Classification 0-1 test loss is **much more benign than regression MSE**; so l2-inductive bias could work better for classification tasks

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- Most theoretical works on benign overfitting focus on linear/kernel setting.
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- We'll discuss recent works in neural networks and open questions.
- Notably: all results on benign overfitting in neural nets require ambient dimension $d \gg n$
- Very unsatisfying: neural nets can be overparameterized in $d \ll n$ regime, when is overfitting benign in this setting?

Which estimators do we care about?

Model	Algorithm	Setting	Estimator
Linear	Gradient descent	Classification	ℓ_2 max-margin
Linear	Gradient descent	Regression	ℓ_2 min-norm interpolator
Linear	Adaboost	Classification	ℓ_1 max-margin
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Neural nets	Gradient descent	Classification	?
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- Next: implicit bias of GD in neural net classification.
- After: “trajectory analysis”, directly analyzing properties of neural nets trained by GD

Telgarsky’13, Soudry-Hoffer-Nacson-Gunasekar-Srebro’18, Ji-Telgarsky’18, …

Implicit bias in neural networks

- Which interpolators do we care about for neural nets?
- We'll focus on classification tasks, training by GD/GF on logistic loss.
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For large class of neural nets, if GD/GF $\theta(t)$ reaches a small enough loss, then $\theta(t)$ converges in direction to a first-order stationary point (KKT point) of the ℓ^2-max margin problem,

$$\min_{\theta} \|\theta\|^2 \quad \text{s.t.} \quad y_i f(x_i; \theta) \geq 1, \, \forall i \in [n]. \tag{1}$$

Implicit bias in neural networks

- Which interpolators do we care about for neural nets?
- We'll focus on classification tasks, training by GD/GF on logistic loss.
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- KKT point does not imply even local optimality in general.
- In general, very little is known about KKT points of (1).

Implicit bias in neural networks

- A setting where we understand KKT points of max-margin: two-layer leaky ReLU nets with nearly-orthogonal data. ($\phi(q) = \max(\gamma q, q)$)

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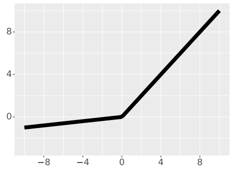

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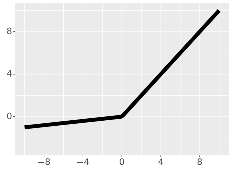

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Suppose data is nearly orthogonal. If θ satisfies KKT conditions for ℓ^2-max-margin, then $\exists s_i > 0$ s.t.

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where $s_i > 0$ satisfy $\max_{i,j} s_i/s_j = O(1)$.

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- Linear model can capture behavior of nonlinear net, trained beyond NTK.

Benign overfitting of neural nets in mixture model

- KKT points for 2-layer leaky nets $\approx \sum_{i=1}^{n} y_i x_i$, when training data is nearly-orthogonal $\left(\|x_i\|^2 \gg n \max_{k \neq j} |\langle x_j, x_k \rangle|\right)$.

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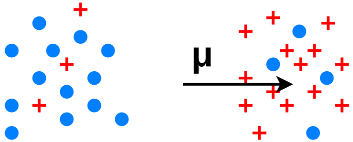

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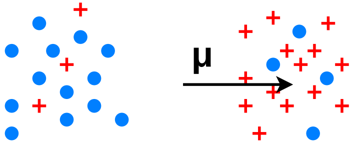

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- Following results will only hold in this low-SNR, high-dimensional regime
  - We'll see consistency is still possible in this setting

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- $\exp(-\Omega(n\|\mu\|^4/d))$ is minimax-optimal!

Benign overfitting of neural nets in mixture model

Recall $\mathrm{sgn}(f(x;\theta)) = \mathrm{sgn}(\langle \sum_{i=1}^n y_i x_i, x\rangle)$. What does this estimator look like? Since $x_i = \tilde{y}_i \mu + z_i$,

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Overfitting component helps interpolation, signal helps generalization:

Training data: classify (x_i, y_i) correctly	Test data: classify $(x, \tilde{y})$ correctly
$\langle y_i x_i, \sum_{i=1}^n \tilde{y}_i z_i\rangle$ is large, positive,	$\langle \tilde{y}x, \sum_{i=1}^n \tilde{y}_i z_i\rangle$ is small, random $\pm$,
$\langle y_i x_i, n\mu\rangle$ is small, noisy labels make $\pm$.	$\langle \tilde{y}x, n\mu\rangle$ is (optimally) large, positive.

- Signal and overfitting component balanced to allow both interpolation + generalization

Other approaches for benign overfitting in neural nets

- Analysis of implicit bias (KKT conditions, minimum norm interpolation, …)

  Frei-Vardi-Bartlett-Srebro'23; Kornowski-Yehudai-Shamir'23; Kou-Chen-Gu'23; …

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- "Trajectory analysis": directly track the weights of neural net trained by GD/GF from random initialization on noisy data, show that it achieves small train and test error Frei-Chatterji-Bartlett'22; Xu-Gu'23; Kou-Chen-Chen-Gu ICML'23; Xu-Wang-Frei-Vardi-Hu'23; Meng-Zou-Cao'23; …
  - Characterizes finite time performance
  - More narrow, less clear "why" benign overfitting happens

Benign overfitting from trajectory analysis

- Directly examine inductive bias of training by GD/GF, e.g. in 2 layer nets

$$f(x;\theta) = \textstyle\sum_{j=1}^{m} a_j \phi(\langle \theta_j, x \rangle), \quad \hat{L}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell\big(f(x_i;\theta)\big),$$

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  - These two must be very different for benign overfitting to occur

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- No dependence on number of neurons in network.
- Benign overfitting if t is large and $n\|\mu\|^4 \gg d$.
- Same generalization bound as KKT analysis, but now holds throughout GD trajectory.
  - Only tolerates $p = O(1)$, rather than $p < {}^{1}\!/_{2}$ from KKT analysis.

Frei-Chatterji-Bartlett'22; Xu-Gu'23

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- Known proofs all rely on nearly-orthogonal data ($d \gg n$) to show this

Chatterji-Long'21; Frei-Chatterji-Bartlett'22

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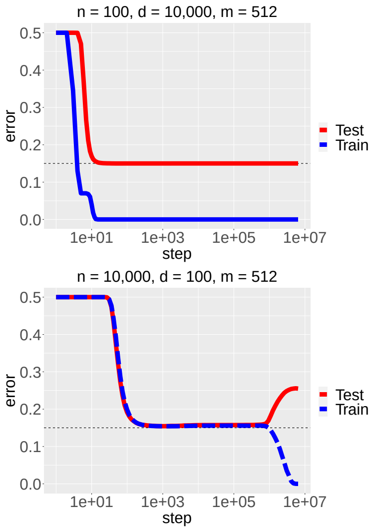
- Learning dynamics different in $n > d$ setting; overfitting less ‘benign’
  $\longrightarrow$ “Blessing of dimensionality”? See also: [Kornowski-Yehudai-Shamir’23]



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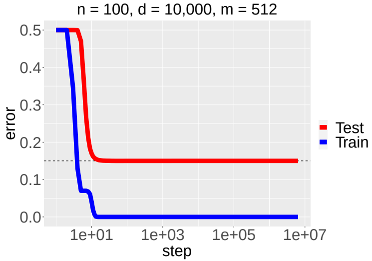
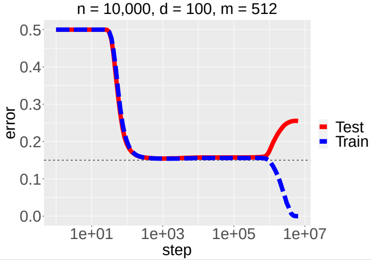
- Neural net trained on high-dimensional mixture model: (provably) benign; low-dimensional: tempered?





Mallinar-Simon-Abedsoltan-Pandit-Belkin-Nakkiran'22

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- Which neural net interpolators do we care about in regression?
- Necessary and sufficient conditions for benign overfitting in linear classification?
  - Fairly complete picture of min-ℓ^2 linear regression, but mostly sufficiency guarantees in classification.
  - Dream: data-dependent, signal-dependent, tight guarantees.

Thanks!