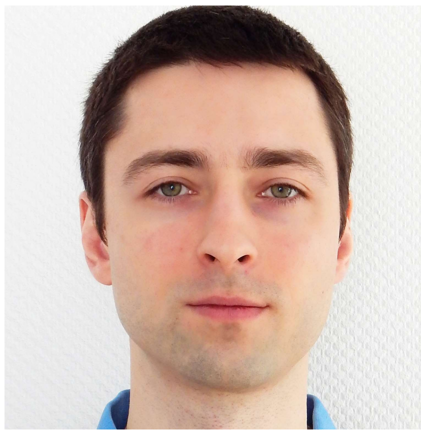
Online Learning under Adversarial Nonlinear Constraints



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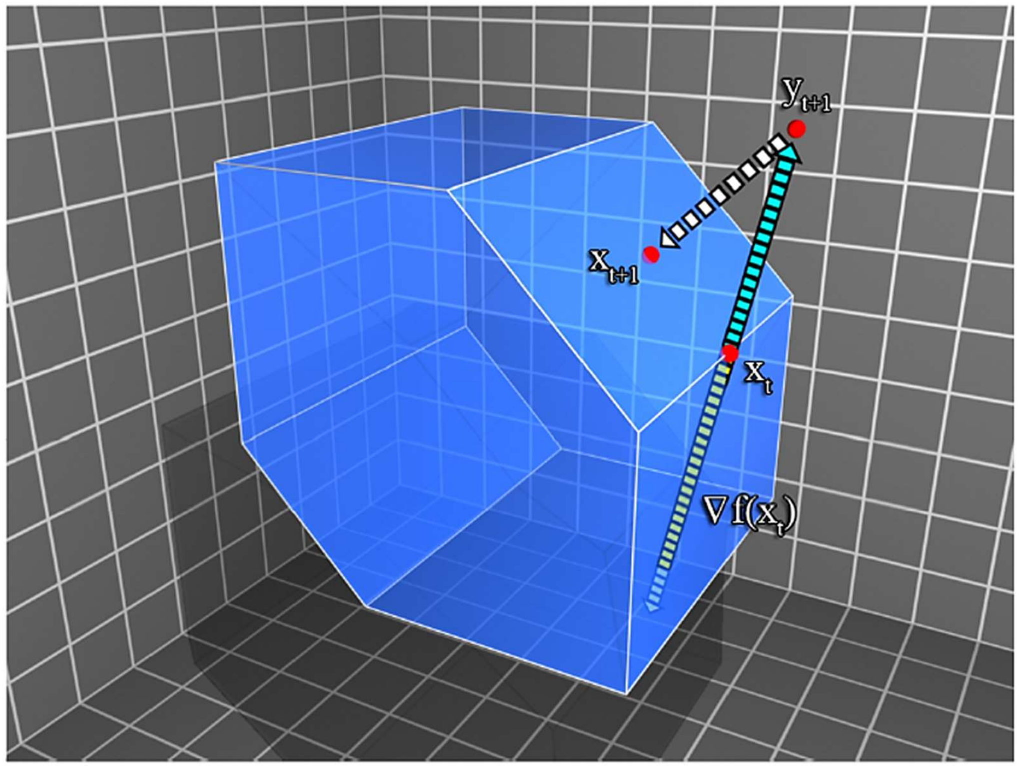
Online Gradient Descent

Zinkevich ICML'03

$$y_{t+1} = x_t - \eta_t \nabla f_t(x_t)$$
$$x_{t+1} = \mathrm{Proj}_{\mathcal{C}}(y_{t+1})$$



Hazan Found. Trends Optim. 2016

Time varying costs $f_t(x_t)$

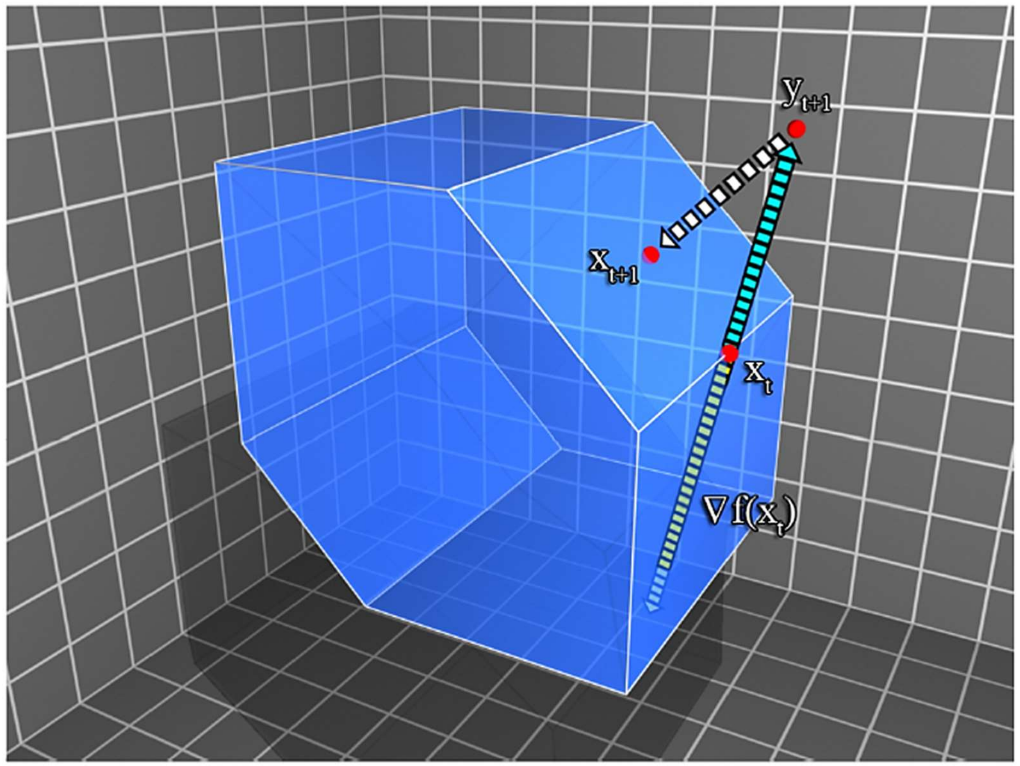
Time invariant (known) constraints $\mathcal{C}$

x_t is feasible (projection)

Online Gradient Descent

Zinkevich ICML'03

$$
\begin{aligned}
y_{t+1} &= x_t - \eta_t \nabla f_t(x_t) \\
x_{t+1} &= \text{Proj}_{\mathcal{C}}(y_{t+1})
\end{aligned}
$$



Hazan Found. Trends Optim. 2016

Time varying costs $f_t(x_t)$

Time invariant (known) constraints $\mathcal{C}$

x_t is feasible (projection)

This Work

Time varying costs $f_t(x_t)$

Time varying (unknown) constraints $\mathcal{C}_t$

$\{x_t\}_{t=1}^T$ converges to feasible $\mathcal{C}_T$

Applications (Time varying constraints)

Sun et al. ICML'17	adversarial contextual bandits
Chen et al. IEEE-TSP'17	network resource allocation
Cao and Liu IEEE-TC'19	logistic regression
Liu et al. SIGMETRICS'22	job scheduling, ridge regression
Castiglioni el al. NeurIPS'22	repeated auctions (internet advertising)

Prior Work

Primal-Dual Methods

Yu et al. NeurIPS'17; Sun et al. ICML'17; Chen et al. IEEE-TSP'17; Neely & Yu arXiv'17

$$\text{Regret}_T = \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x^\star)$$

$$\text{s.t.} \quad \sum_{t=1}^T g_t(x_t) \geq -c\sqrt{T}$$

Online Iterates

$$x^\star \text{ such that } g_t(x^\star) \geq 0 \quad \forall t$$

Benchmark (Optimal Solution)

Prior Work

Primal-Dual Methods

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$$\text{Regret}_T = \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x^\star)$$

$$\text{s.t.} \quad \sum_{t=1}^T g_t(x_t) \geq -c\sqrt{T}$$

Weaker $\begin{cases} \exists x_t & g_t(x_t) \gg 0 \\ \forall \ell \neq t & g_\ell(x_\ell) < 0 \end{cases}$

$$x^\star \text{ such that } g_t(x^\star) \geq 0 \quad \underline{\forall t}$$

Asymmetric comparison

Prior Work

Primal-Dual Methods

Yu et al. NeurIPS'17; Sun et al. ICML'17; Chen et al. IEEE-TSP'17; Neely & Yu arXiv'17

$$\text{Regret}_T = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^\star)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} g_t(x_t) \geq -c\sqrt{T}$$

$$x^\star \text{ such that } g_t(x^\star) \geq 0 \quad \underline{\forall t}$$

This Work

$$\text{s.t.} \quad g_T(x_T) \geq -c/\sqrt{T}$$

$$x^\star \text{ such that } g_T(x^\star) \geq 0$$

Our Problem Formulation

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x^\star \in \mathcal{C}_T} \sum_{t=1}^{T} f_t(x^\star) \quad \text{subject to} \quad g_T(x_T) \geq -\frac{c}{\sqrt{T}}$$

where $\quad \mathcal{C}_T = \{x \in \mathbb{R}^n : g_T(x) \geq 0\}$

Assumptions:

* Standard Convexity & Compactness

* Time Varying Constraints

Assumptions: Time Varying

(1) Slowly Time Varying Constraints $\|g_{t+1} - g_t\|_\infty \le O(1/t)$

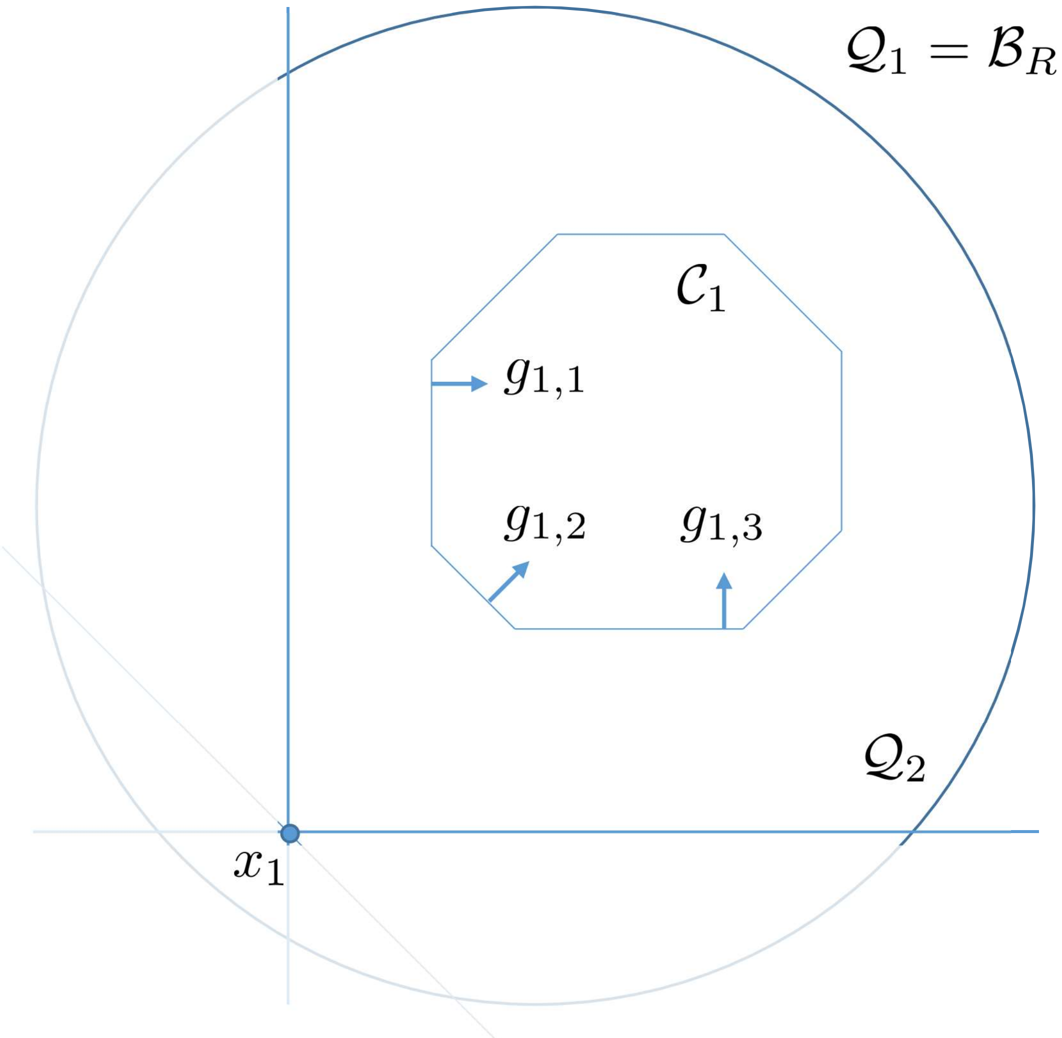
Ensures $g_T(x_T) \ge -\frac{c}{\sqrt{T}}$

(2) Geometric: feasible set $\mathcal{C}_t \subseteq \mathcal{Q}_t$ cone intersection

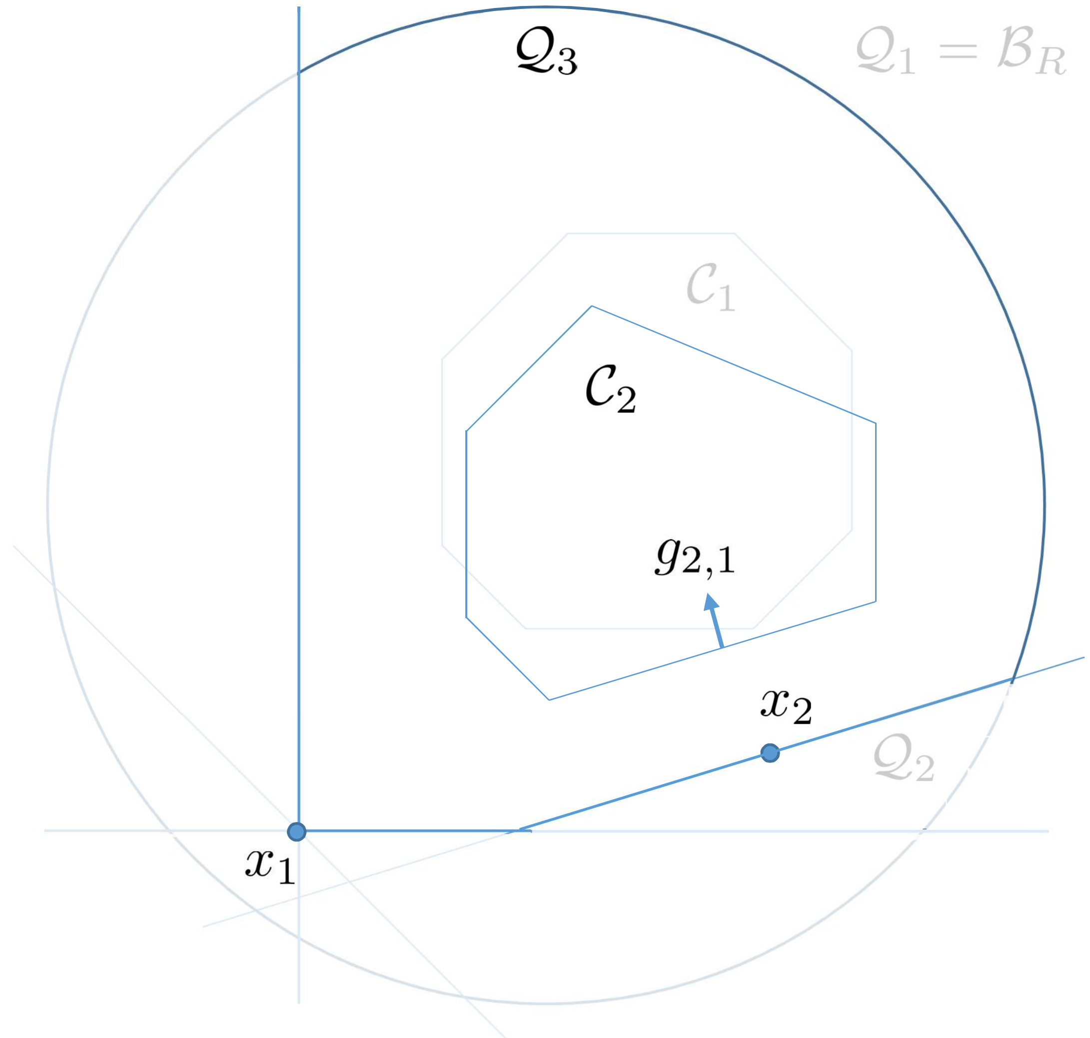
Ensures $\text{Regret} \le O(\sqrt{T})$

Cone Intersection

Linear $g_{t,i}(x) = g_{t,i}^\top x$



$\mathcal{C}_\ell \subseteq \mathcal{Q}_2$ for all $\ell \geq 2$



$\mathcal{C}_\ell \subseteq \mathcal{Q}_3$ for all $\ell \geq 3$

Constraint Violation Velocity Projection (CVV-Pro)

CVV-Pro

initialize: $\alpha > 0$, $\{\eta_t = \frac{1}{\alpha\sqrt{t}}\}_{t\geq 1}$

for $t = 1$ **to** T **do**

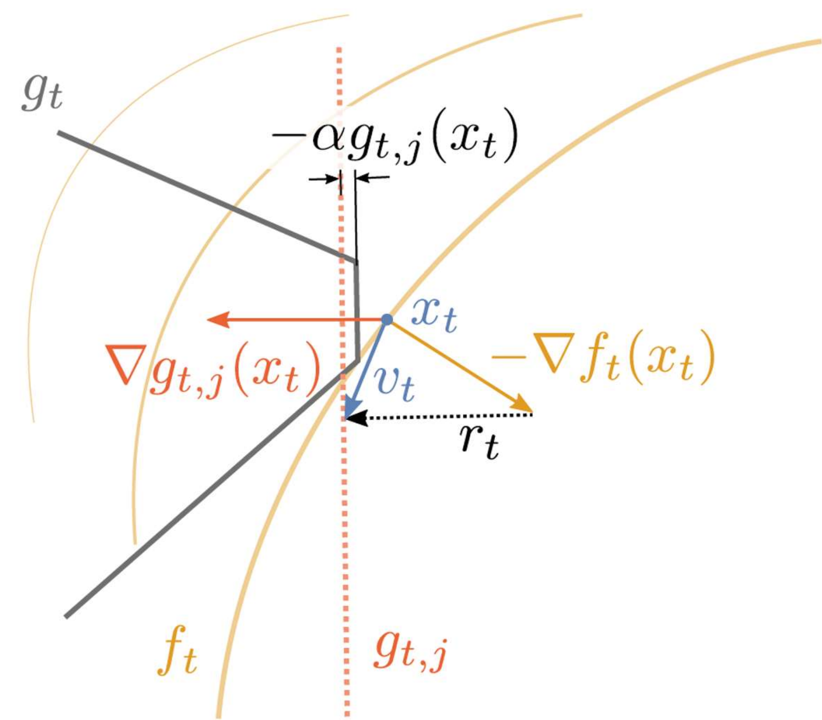
play x_t

observe: $f_t(x_t), \nabla f_t(x_t)$ and $\{(g_i(x_t), \nabla g_i(x_t))\}_{i\in I(x_t)}$

solve velocity projection problem

$$v_t = \arg\min_{v\in V_\alpha(x_t)} \frac{1}{2}\|v + \nabla f_t(x_t)\|^2$$

update $x_{t+1} = x_t + \eta_t v_t$



Inspired by Muehlebach & Jordan JMLR'22

local linear information
for all violated constraints

Constrained Gradient Flow $\dot{x}(t)^+ = -\nabla f(x(t)) + R(t)$

Velocity Polyhedron $V_\alpha(x_t) := \{v \in \mathbb{R}^n \mid [\nabla g_i(x_t)]^\top v \geq -\alpha g_i(x_t),\ \forall i \in I(x_t)\}$

Main Result

Theorem 1. Given $R, L_{\mathcal{F}}, x_1 \in \mathcal{B}_R$

Let $\alpha = L_{\mathcal{F}}/R$, $\eta_t = 1/(\alpha\sqrt{t+15})$. Then for all $T \geq 1$

(regret) $\sum_{t=1}^{T} f_t(x_t) - \min_{x^\star \in \mathcal{C}_T} \sum_{t=1}^{T} f_t(x^\star) \leq 246 L_{\mathcal{F}} R\sqrt{T}$;

(feasibility) $g_{t,i}(x_t) \geq -265 \left[\frac{L_{\mathcal{G}}}{R} + 4\beta_{\mathcal{G}}\right] \frac{R^2}{\sqrt{t+15}}$, for all $t \in [T]$, $i \in [m]$;

Summary (CVV-Pro)

1. Handles **unknown** & **time-varying** constraints
2. New type of Oracle
   a) **local** information for all violated constraints
   b) efficient projection of $-\nabla f_t(x_t)$ onto $V_\alpha(x_t)$
   c) $V_\alpha(x_t)$ is **sparse linear** velocity polyhedron
3. Guarantees:
   a) **optimal** $\sqrt{T}$ regret
   b) x_T **converges** to feasible set with rate $1/\sqrt{T}$

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x^\star \in \mathcal{C}_T} \sum_{t=1}^{T} f_t(x^\star)$$

$$\text{subject to} \quad g_T(x_T) \geq -\frac{c}{\sqrt{T}}$$