A Classification of *G*-Invariant Shallow Neural Networks

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The problem

- Want to fit a neural network (NN) to a G-invariant target function f where G is a group:

$$f(gx) = f(x) \forall g \in G, x \in \text{dom}(f).$$

- **G-invariant neural architecture design:**
  - What is the best way to constrain the NN to be G-invariant?
- **G-invariant neural architecture search (G-NAS):**
  - What are all the possible G-invariant architectures?
  - How are they connected in the search space?

Key contributions

- **Scope:**
  - Single-hidden-layer or "shallow neural networks" (SNNs) with ReLU activation.
  - Finite orthogonal matrix groups G.
- Two theorems characterizing G-invariant architecture space (see preprint[1] for details):
  - **Theorem 4:** Classifies all G-invariant SNNs (G-SNNs) in terms of "signed permutation representations".
  - **Theorem 5:** Characterizes the "network morphisms" that define the connectivity structure of G-invariant architecture space.

[1] https://arxiv.org/abs/2205.09219

Illustrative example

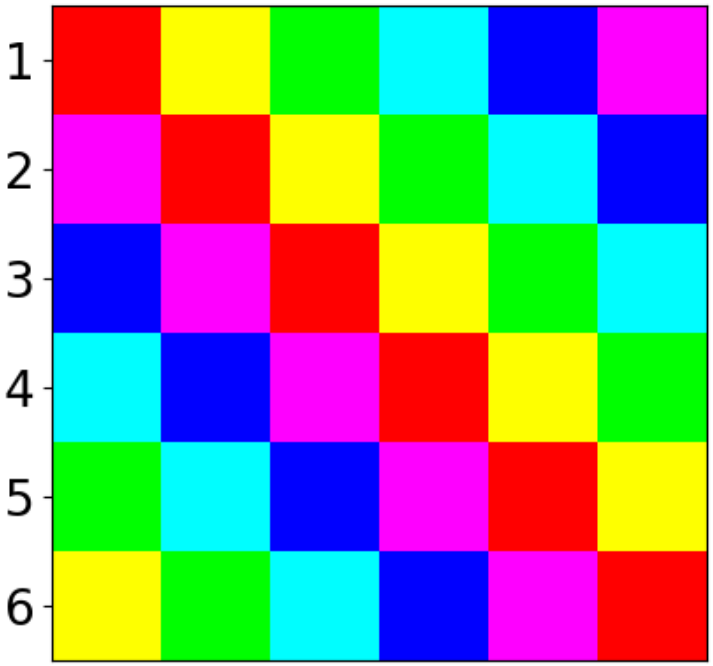
- Target function $f : \mathbb{R}^6 \mapsto \mathbb{R}$ invariant under the group G of all cyclic permutations of the input dimensions:

$$f(x_1, \ldots, x_6) = f(x_{\pi(1)}, \ldots, f_{\pi(6)}) \forall \pi \in G.$$

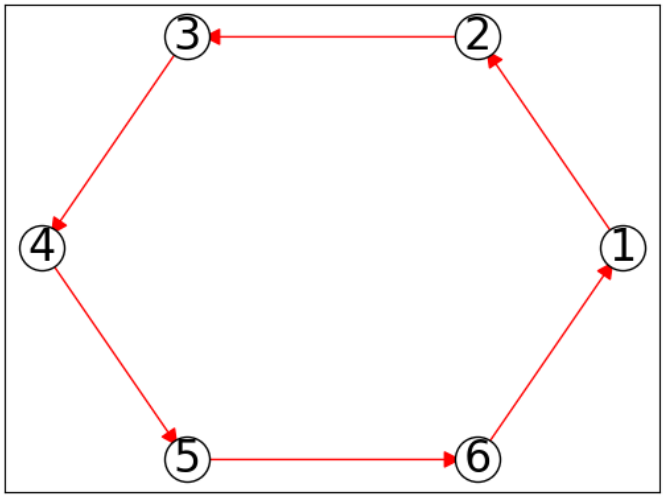
- Six *irreducible* G-SNN architectures.
- Every G-SNN architecture is a sum of irreducibles.

Irreducible architecture 1

Weight pattern

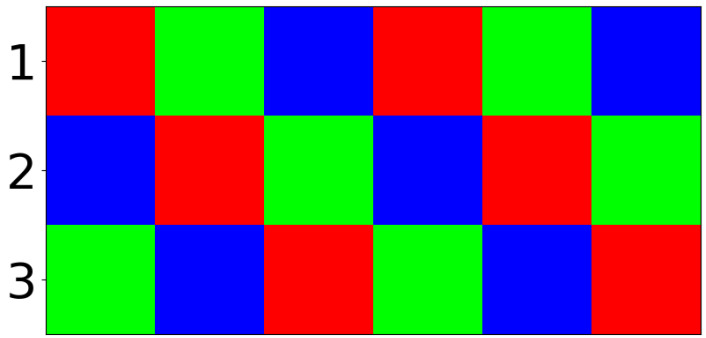


Cohomology class



Irreducible architecture 2

Weight pattern



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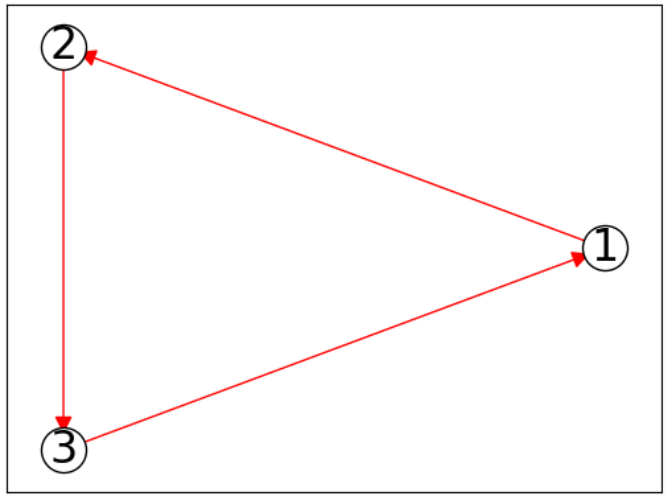
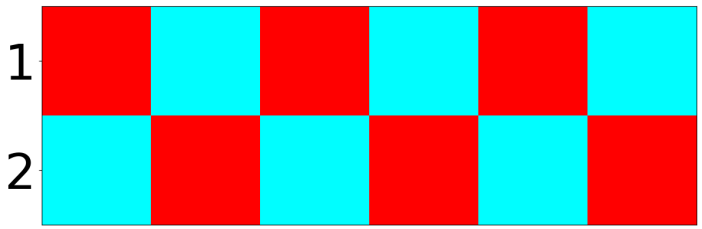


Irreducible architecture 3

Weight pattern



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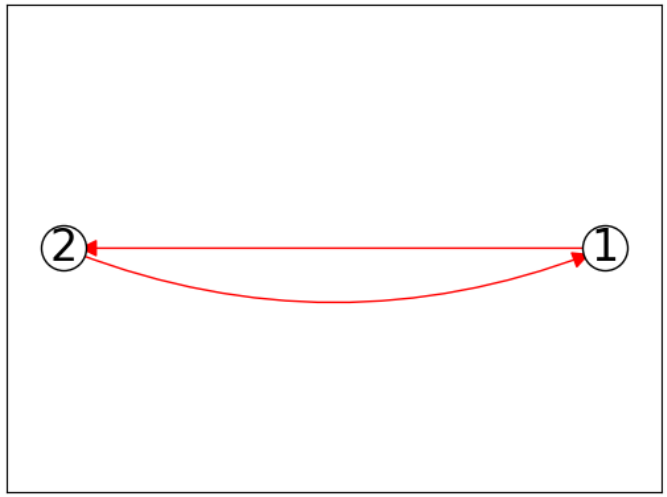
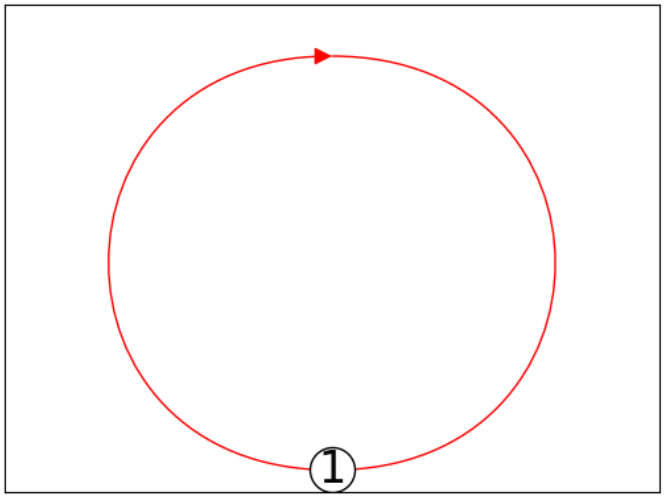


Irreducible architecture 4

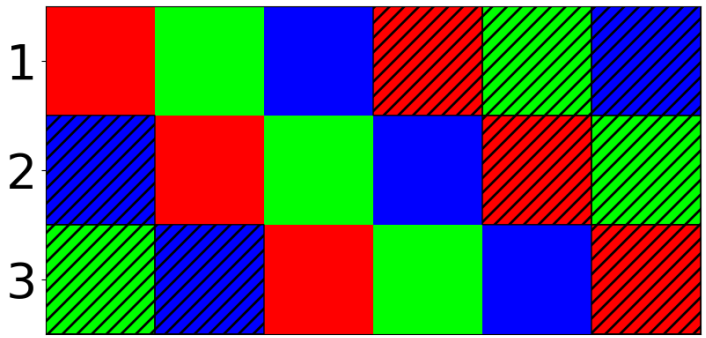
Weight pattern

Cohomology class



Irreducible architecture 5

Weight pattern



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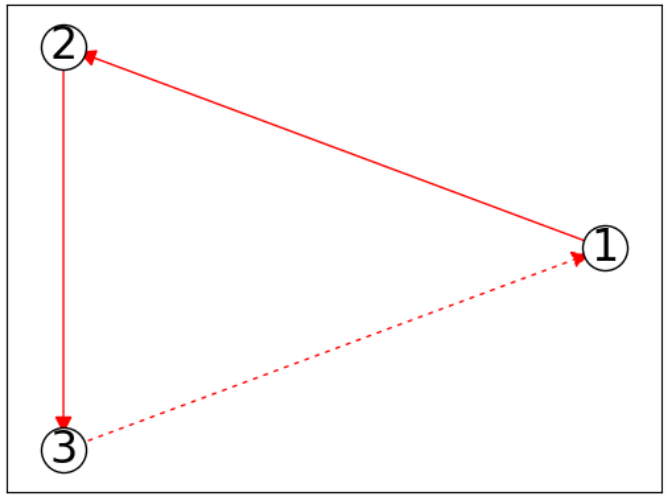
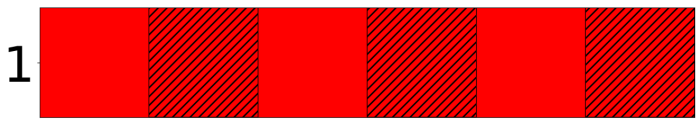


Irreducible architecture 6

Weight pattern



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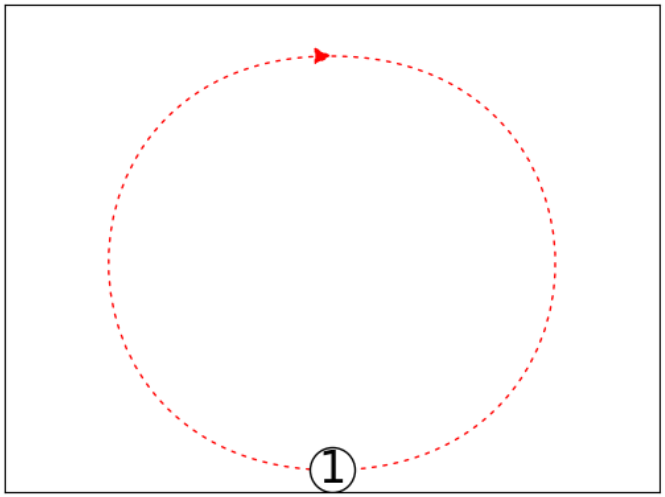
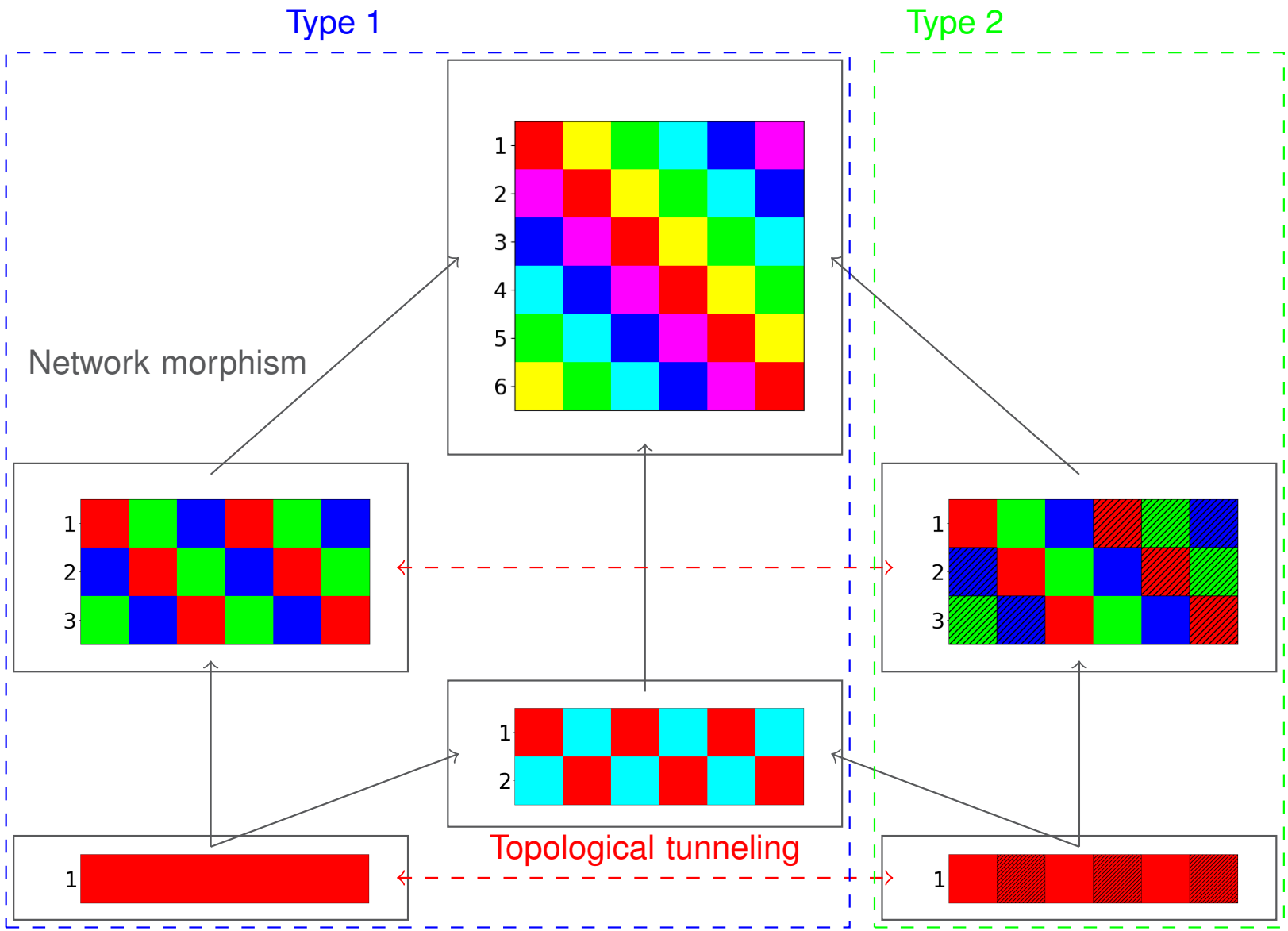


Irreducible architecture space



Conclusion and next steps

- First step towards G-invariant neural architecture design.
- This work characterizes structure of G-invariant architecture space for SNNs.
- **Next steps:**
  - Extend theory to deep NNs.
  - Implement G-NAS.

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