Sparse Hypergraph Community Detection Thresholds in Stochastic Block Model

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Contributions

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- We confirm the positive part of the conjecture, the possibility of non-trivial reconstruction above the threshold, for the case of two blocks by comparing the hypergraph stochastic block model with its Erdös-Rényi counterpart.
- We show the negative part of the conjecture by relating the model with the so-called *multi-type Galton-Watson hypertrees* and considering the broadcasting problem on these hypertrees.

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- HSBM (denoted by $\mathcal{H}_d(n, p_n, q_n)$): Let $H = ([n], \mathcal{E})$ be a d-uniform hypergraph with vertex set $[n] := \{1, 2, \cdots, n\}$ and hyperedge set $\mathcal{E}$, $\sigma := (\sigma_1, \cdots, \sigma_n) = \{+1, -1\}^n$ the spins on $[n]$. Let $\mathcal{S}_n$ be the set of all pairs (H, σ), we can generate a random pair (H, σ) from the finite set $\mathcal{S}_n$ as follows:

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    - First generate i.i.d random variables $\sigma_i \in \{+1, -1\}$ uniformly for each $i \in [n]$.
    - Then, for the obtained $\sigma = (\sigma_1, \cdots, \sigma_n)$, we generate a random d-uniform hypergraph H where an hyperedge $e = \{i_1, \cdots, i_d\}$ is included independently with probability p_n if $\sigma_{i_1} = \cdots = \sigma_{i_d}$, and with probability q_n otherwise, where $0 < q_n < p_n < 1$ (p_n, q_n possibly depending on n).

- Suppose $\mathcal{C}_1 = \{i \in [n] | \sigma_i = +1\}$ and $\mathcal{C}_2 = \{i \in [n] | \sigma_i = -1\}$ are two communities in the hypergraph H. The goal of community detection is to estimate the unknown spin σ up a sign flip by observing H only from a sample (H, σ) drawn from $\mathcal{H}_d(n, p_n, q_n)$.

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- For SBM, Decelle et al. (2011) conjectured the existence of a sharp threshold (called *Kesten-Stigun threshold*): if $p_n = \frac{a}{n}$, $q_n = \frac{b}{n}$, then detection is possible if and only if $(a-b)^2 > 2(a+b)$.

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  - Proof/Algorithm: Pal and Zhu (2021), Stephan and Zhu (2022)

Main Results

- Let $\mathcal{H}_d(n, \frac{p_n+(2^{d-1}-1)q_n}{2^{d-1}})$ be the Erdös-Rényi model in which each hyperedge is included with a common probability $\frac{p_n+(2^{d-1}-1)q_n}{2^{d-1}}$. Let $\mathbb{P}_n$ and $\tilde{\mathbb{P}}_n$ denote the probability measures with respect to $\mathcal{H}_d(n, p_n, q_n)$ and $\mathcal{H}_d(n, \frac{p_n+(2^{d-1}-1)q_n}{2^{d-1}})$, respectively.

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- **Theorem 1** If $\beta^2 > \alpha$, then $\mathbb{P}_n$ and $\tilde{\mathbb{P}}_n$ are asymptotically orthogonal. Let X_{ζ_n} be the number of loose cycles of length ζ_n and define

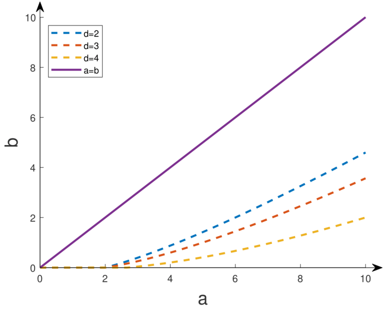
$$\hat{\alpha}_n := \frac{d|\mathcal{E}|}{\binom{n}{d-1}}, \qquad \hat{\beta}_n := (2\zeta_n X_{\zeta_n} - \hat{\alpha}_n^{\zeta_n})^{\frac{1}{\zeta_n}},$$

where $\zeta_n = \lfloor \log^{1/4} n \rfloor$, $|\mathcal{E}|$ is the number of observed hyperedges, then $\hat{a}_n = \frac{1}{d-1}(\hat{\alpha}_n + (2^{d-1}-1)\hat{\beta}_n)$ and $\hat{b}_n = \frac{1}{d-1}(\hat{\alpha}_n - \hat{\beta}_n)$ are consistent estimators for a and b, respectively.

- **Theorem 2**[1] If $\beta^2 < \alpha$, then for any fixed vertices v_1 and v_2, $H \sim \mathcal{H}_d(n, p_n, q_n)$,

$$\lim_{n\to\infty} \mathbb{P}_n(\sigma_{v_1} = +1 | H, \sigma_{v_2}) = \frac{1}{2}.$$

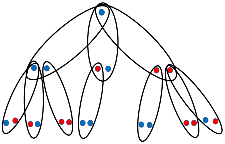
$\mathbb{P}_n$ and $\tilde{\mathbb{P}}_n$ are mutually contiguous. Further more, there is no consistent estimator for a and b.



[1]This theorem becomes suspicious now since Ludovic Stephan and Yizhe Zhu pointed out a key mistake in our proofs.

Building Blocks

- Let X_{ζ_n} be the number of ζ_n-loose cycle of a hypergraph H. Suppose $\zeta_n = \mathcal{O}(\log^{1/4}(n))$.



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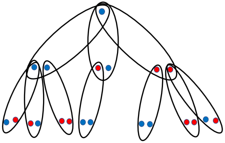
    - If $H \sim \mathcal{H}_d(n, p_n, q_n)$, then $X_{\zeta_n} \xrightarrow{d} \text{Pois}(\frac{\alpha^{\zeta_n}+\beta^{\zeta_n}}{2\zeta_n})$;



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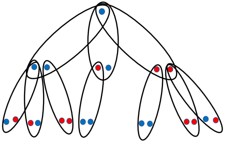

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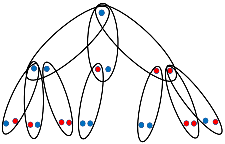
- Suppose (T, ρ, τ) is a multi-type Galton-Watson hypertree where the offspring distribution has mean $\alpha > 1$, if $\beta^2 < \alpha$, then

$$\lim_{l\to\infty} \mathbb{P}(\tau_\rho = +1|\tau_{\partial T_l}) = \frac{1}{2} \quad \text{a.s.}$$

where $\tau_{\partial T_l} = \{\tau_v | v \in \partial T_l\}$.



- **Theorem 5.2 (Pal and Zhu, 2021)** Let $(H, \rho, \sigma)_l$ be the rooted hypergraph (H, ρ, σ) truncated at generation l from ρ, $(T, \rho, \tau)_l$ the rooted hypertree (T, ρ, τ) truncated at generation l from ρ, then for sufficiently large n, $l = c\log(n)$ with $c\log(\alpha) < \frac{1}{4}$ and c is a constant, there exists a coupling between (H, ρ, σ) and (T, ρ, τ) such that $(H, \rho, \sigma)_l \equiv (T, \rho, \tau)_l$ with probability at least $1 - n^{-1/5}$.

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- Let V_1, V_2, V_3 be a random partition of the vertex set $\mathcal{V}(H)$ such that V_2 separates V_1 and V_3. If $|V_1 \cup V_2| = o(\sqrt{n})$ for a.a.e H, then

$$\mathbb{P}(\sigma_{V_1}|H, \sigma_{V_2}) = (1 + o(1))\mathbb{P}(\sigma_{V_1}|H, \sigma_{V_2 \cup V_3})$$

for a.a.e. H and σ.

- **Theorem 4.1 (Wormald et al., 1999)** Let $\mathbb{P}_n$ and $\tilde{\mathbb{P}}_n$ be two fixed sequences of probability measures on a common measurable space, $Y_n = \frac{\mathbb{P}_n}{\tilde{\mathbb{P}}_n}$ the density of $\mathbb{P}_n$ with respect to $\tilde{\mathbb{P}}_n$. For $i \geq 1$, let $\lambda_i > 0$, $\delta_i \geq -1$, for each n, suppose random variables X_{in} satisfy

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  - For every non-negative integers $s_1, \cdots, s_k$,

$$\tilde{\mathbb{E}}(Y_n[X_{1n}]_{s_1} \cdots [X_{kn}]_{s_k})/\tilde{\mathbb{E}} Y_n \to \prod_{i=1}^{k} (\lambda_i(1+\delta_i))^{s_i}.$$

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  - $$\sum_{i \geq 1} \lambda_i \delta_i^2 < \infty;$$
  - $$\tilde{\mathbb{E}} Y_n^2 / (\tilde{\mathbb{E}} Y_n)^2 \to \exp(\sum_{i \geq 1} \lambda_i \delta_i^2).$$

  Then, $\tilde{\mathbb{P}}_n$ and $\mathbb{P}_n$ are contiguous.

Conclusion

We prove a conjecture on the community detection thresholds in the HSBM with two blocks where the hypergraph is uniform and sparse.

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