Necessary and sufficient graphical conditions for optimal adjustment sets in causal graphical models with hidden variables (#3495)

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October 18, 2021

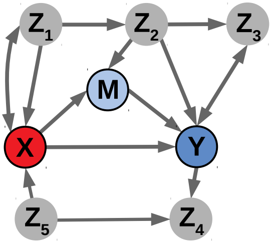
DLR Institute of Data Science and TU Berlin

Knowledge for Tomorrow

Causal inference preliminaries

Task Given a qualitative causal graph and data, estimate causal effect of X on Y [Pearl, 2009]:

$$p(Y \mid do(X = x))$$



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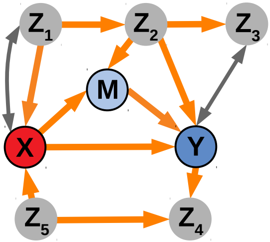
Graph type Acyclic directed mixed graph (ADMG) $\mathcal{G} = (\mathbf{V}, \mathcal{E})$ with **directed (**$\rightarrow$**)** and *bi-directed* ($\leftrightarrow$) edges representing arbitrary latent confounders



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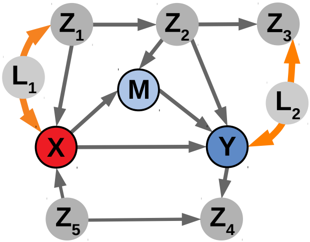

Causal inference preliminaries

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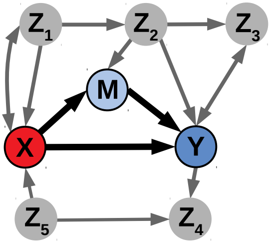
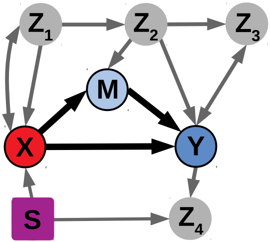
Different types of effects Here total causal effect through direct and indirect path through mediator(s) M



Causal inference preliminaries

Extended task Given a qualitative causal graph and data: Estimate *conditional causal effect* of X on Y given S

$$p(Y \mid do(X = x), S = s)$$

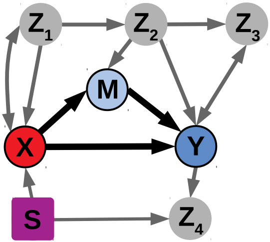


Causal inference preliminaries

Identifiability Effect is *identifiable* if it can be expressed as a function of the observational distribution $p(\mathbf{V})$ [Pearl, 2009]:

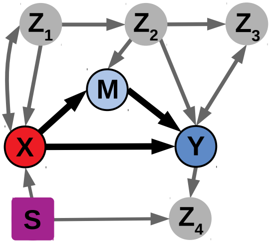
$$p(Y \mid do(X = x), S = s) = q(p(\mathbf{V}))$$

Different approaches: **Backdoor adjustment** / Frontdoor adjustment / General do-calculus



Valid backdoor adjustment sets A set $\mathbf{Z}$ for the total causal effect of X on Y is called *valid* relative to (X, Y) if the interventional distribution for setting $do(X = x)$ factorizes as:

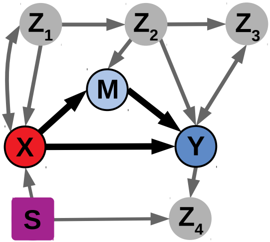
$$p(Y|do(X = x)) = q(p(\mathbf{V})) = \int_{\mathbf{Z}} p(Y|x, \mathbf{z})p(\mathbf{z})d\mathbf{z}$$



Causal inference preliminaries

Generalized backdoor criterion [Perković et al., 2018]: With $\mathbf{forb}(X, Y) = X \cup des(Y\mathbf{M})$ a set $\mathbf{Z}$ is valid if:

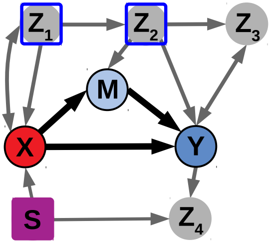
1. $\mathbf{Z} \cap \mathbf{forb} = \emptyset$, and
2. all proper non-causal paths from X to Y are blocked by $\mathbf{Z}$.



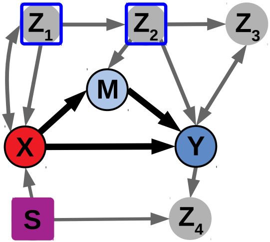
Causal inference preliminaries

Adjust-set [Perković et al., 2018] is valid if and only if a valid set exists:

$$\textbf{vancs}(X, Y, \mathbf{S}) = an(XY\mathbf{S}) \setminus \textbf{forb} \quad (1)$$



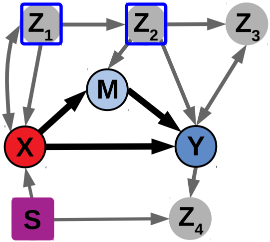
Causal inference preliminaries

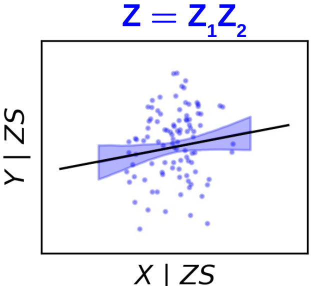


Causal inference preliminaries

(Linear) total causal effect for $x = x' + 1$ with a valid set $\mathbf{Z}$ is equal to $\beta_{YX\cdot\mathbf{ZS}}$ in

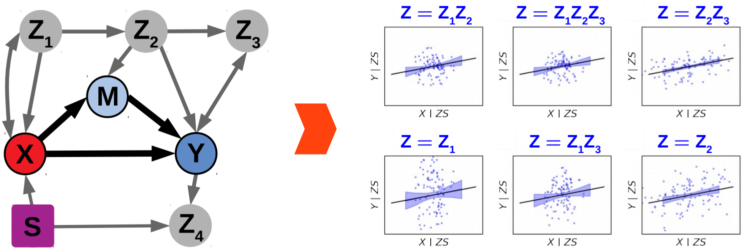
$$Y = \boxed{\beta_{YX\cdot\mathbf{ZS}}} X + \sum_i \beta_{YZ_i\cdot X\mathbf{S}} Z_i + \sum_i \beta_{YS_i\cdot X\mathbf{Z}} S_i \tag{1}$$





Problem setting

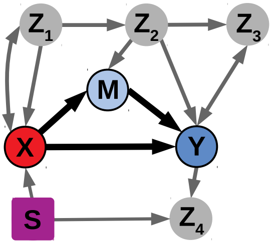
Consider all adjustment sets

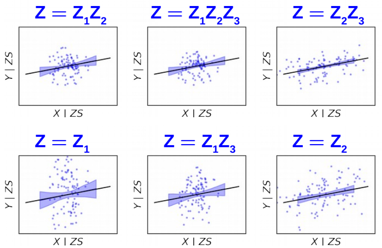


Problem setting

Consider all adjustment sets

All valid sets lead to estimates with zero bias of $\widehat{\beta}_{YX\cdot\mathbf{ZS}}$, but variance strongly differs.

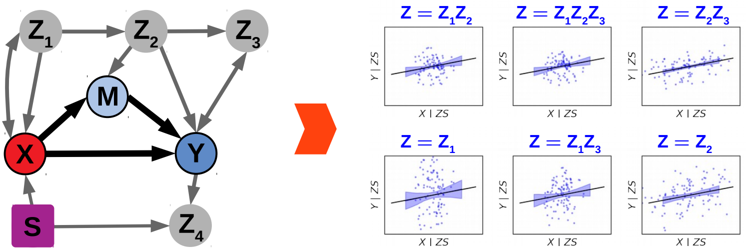




Problem setting

Open problem Find valid adjustment set that yields minimal *asymptotic* variance:

$$\mathbf{Z}_{\text{optimal}} \in \operatorname{argmin}_{\mathbf{Z} \in \mathcal{Z}} E[(\Delta_{yxx'|\mathbf{s}} - \widehat{\Delta}_{yxx'|\mathbf{s}.\mathbf{z}})^2]\,. \qquad (2)$$



Information-theoretic optimal adjustment theory

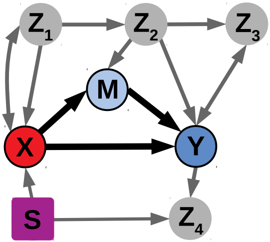
Def.: Conditional mutual information (CMI) for Shannon entropy

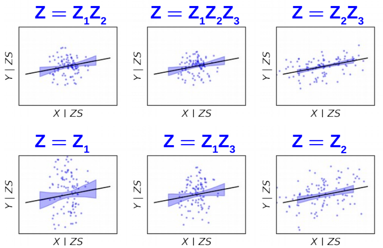
$H_{Y|X} = -\int_{x,y} p(x,y) \ln p(y|x) dxdy$

$$I_{X;Y|Z} \equiv H_{Y|Z} - H_{Y|ZX} \quad (3)$$

$$\geq 0 \quad (4)$$

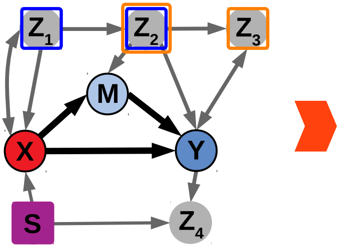
$$= 0 \quad \Leftrightarrow \quad X \perp\!\!\!\perp Y \mid Z \quad (5)$$

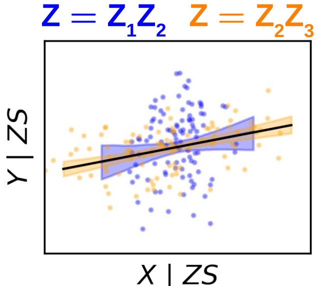




Information-theoretic optimal adjustment theory

Compare Adjust set $\mathbf{Z} = Z_1 Z_2$ vs $\mathbf{O} = Z_2 Z_3$



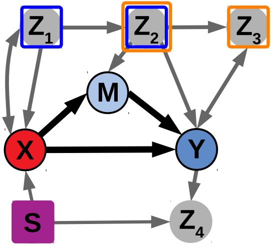


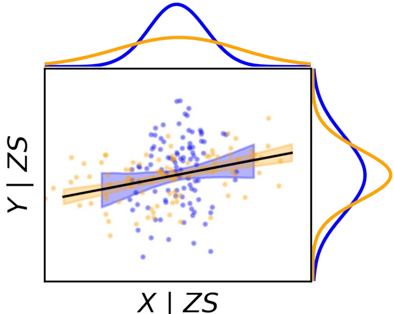
Information-theoretic optimal adjustment theory

Compare Adjust set Z $= Z_1 Z_2$ **vs O** $= Z_2 Z_3$

Two reasons for smaller estimator variance:

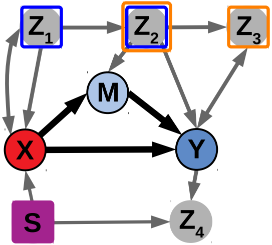
1. **Larger** residual variance of X
2. **Smaller** residual variance of Y

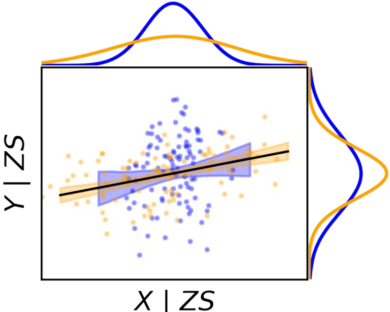




Information-theoretic optimal adjustment theory

Intuition Choose an adjustment set **Z** that maximally constrains Y and minimally constrains X



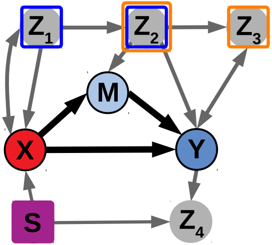


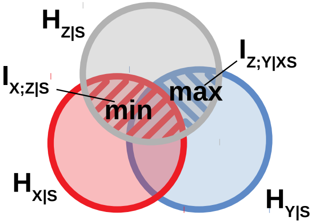
Information-theoretic optimal adjustment theory

Intuition Choose an adjustment set $\mathbf{Z}$ that maximally constrains Y and minimally constrains X

Def. 1: Adjustment information

$$J_{\mathbf{Z}} \equiv J_{XY|\mathbf{S}.\mathbf{Z}} \equiv I_{\mathbf{Z};Y|X\mathbf{S}} - I_{X;\mathbf{Z}|\mathbf{S}} \tag{3}$$



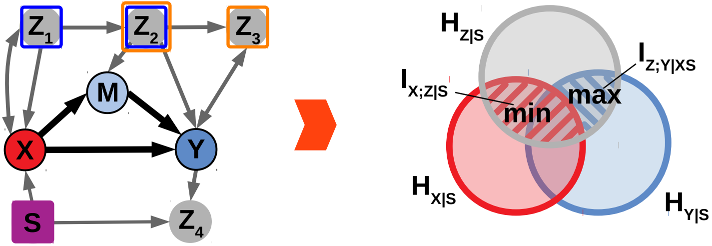


Information-theoretic optimal adjustment theory

Optimality results are valid for estimators $\widehat{\Delta}_{yxx'|\mathbf{s.z}}$ that obey

$$\mathbf{Z}_{\text{optimal}} \in \operatorname{argmax}_{\mathbf{Z}\in\mathcal{Z}} J_{\mathbf{Z}} \;\Rightarrow\; Var(\widehat{\Delta}_{yxx'|\mathbf{s.z}_{\text{optimal}}}) = \min_{\mathbf{Z}\in\mathcal{Z}} Var(\widehat{\Delta}_{yxx'|\mathbf{s.z}})$$

In paper theoretically shown for **OLS**, experimentally also for other estimators.



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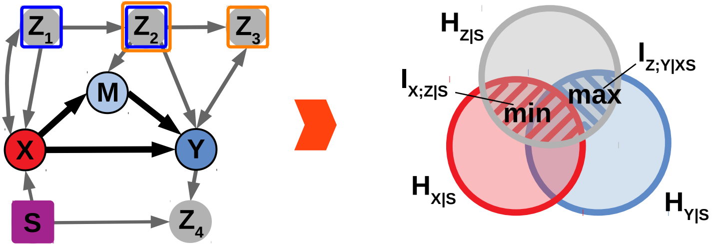

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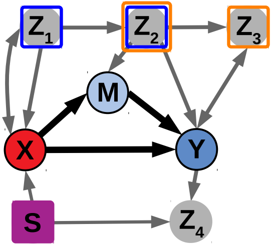
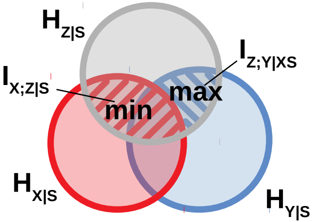
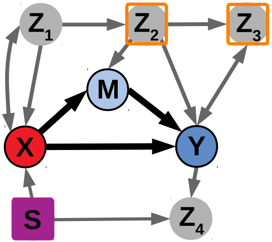
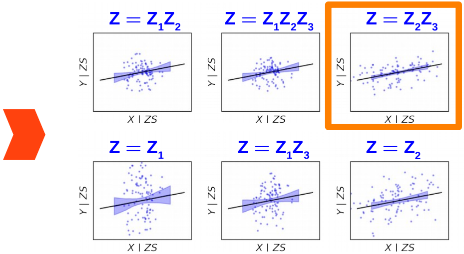




Information-theoretic optimal adjustment theory

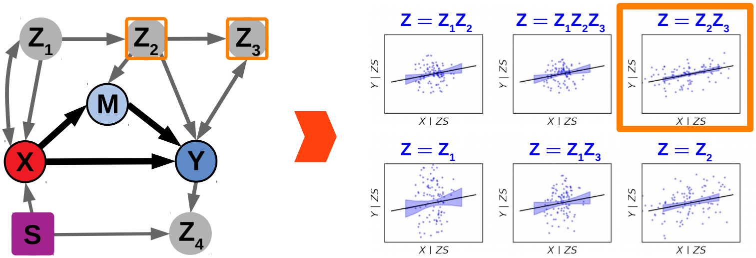
Def. 2: Graphical optimality For a tuple $(\mathcal{G}, X, Y, S)$ *graphical optimality holds* if there is a $\mathbf{Z} \in \mathcal{Z}$ s.t. for all other $\mathbf{Z}' \neq \mathbf{Z} \in \mathcal{Z}$ and *all* distributions $\mathcal{P}$ consistent with $\mathcal{G}$ we have $J_{\mathbf{Z}} \geq J_{\mathbf{Z}'}$.





Information-theoretic optimal adjustment theory

Is there always an optimal adjustment set?



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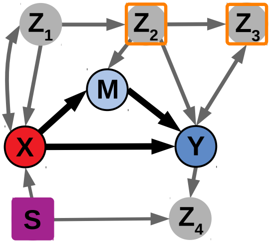

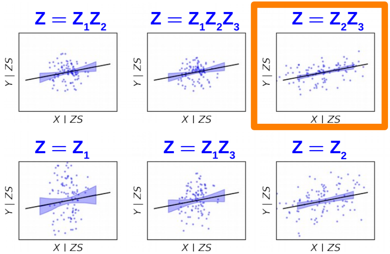

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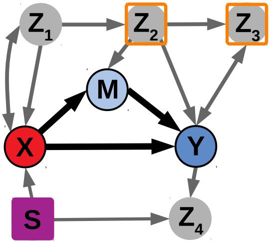
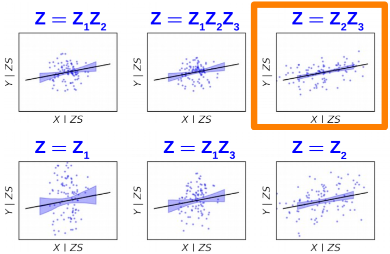


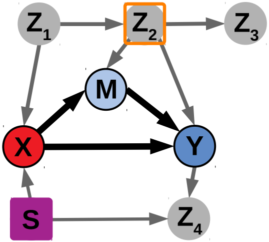


Information-theoretic optimal adjustment theory

Yes for DAGs without hidden variables

([Henckel et al., 2019, Witte et al., 2020, Rotnitzky and Smucler, 2019]):

$$\mathbf{O} = \mathbf{P} = pa(Y\mathbf{M}) \setminus \mathbf{forb}\,. \quad (3)$$



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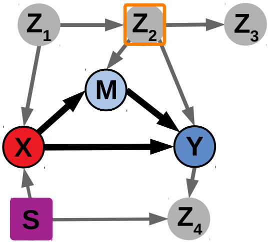

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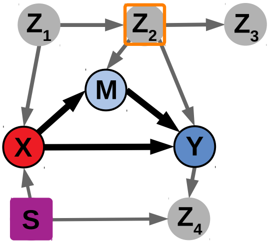
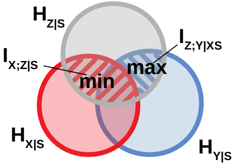
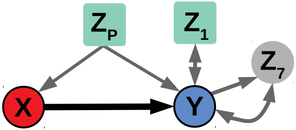


The optimal adjustment set for ADMGs with hidden variables

Intuition Constraining Y by $pa(Y\mathbf{M})$ not enough...

...add spouses since $I(Z_1; Y) > 0$ (as long as $\notin$ **forb**)...





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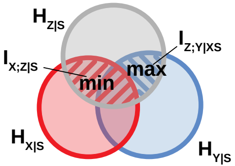

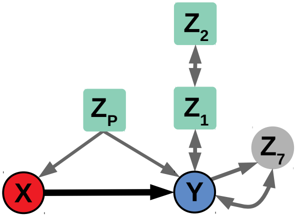

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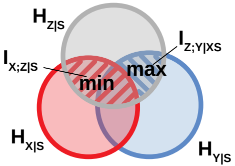

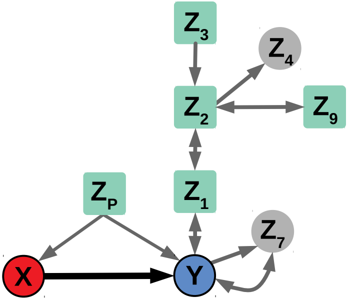

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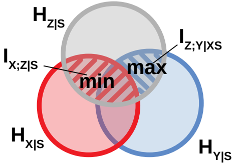

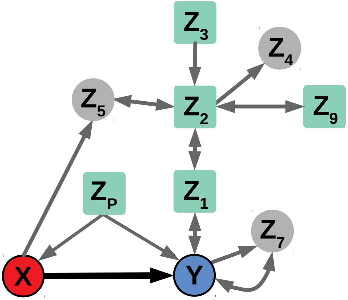

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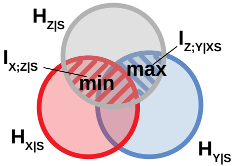
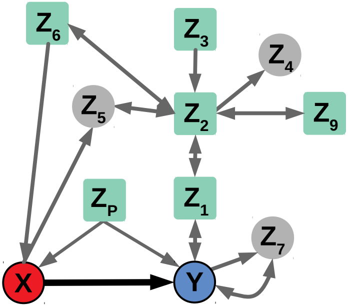
...until a tail is reached or the path ends...

...exclude collider if $C \not\perp\!\!\!\perp X \mid \mathbf{vancs}$ (avoids non-causal paths)...

...except if $C \in \mathbf{vancs}$ where $\mathbf{vancs} = an(XY\mathbf{S}) \setminus \mathbf{forb}$





The optimal adjustment set for ADMGs with hidden variables

Def. O-set: $\mathbf{O}(X, Y, \mathbf{S}) = \mathbf{P} \cup \mathbf{C} \cup \mathbf{P_C}$ where

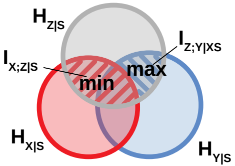
$$\mathbf{P} = pa(Y\mathbf{M}) \setminus \mathbf{forb}$$

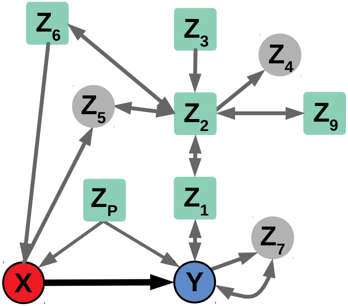
$$\mathbf{C} = \text{“valid collider paths from } W \in Y\mathbf{M}\text{”}$$

$$\mathbf{P_C} = pa(\mathbf{C})$$

where colliders $C \in \mathbf{C}$ fulfill

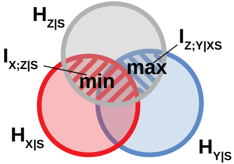
$$(1)\ C \notin \mathbf{forb}, \quad \text{and} \quad (2a)\ C \in \mathbf{vancs} \text{ or } (2b)\ C \perp\!\!\!\perp X \mid \mathbf{vancs}\,. \qquad (4)$$

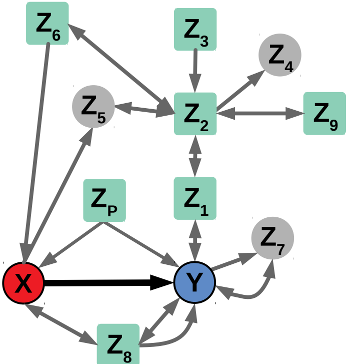




The optimal adjustment set for ADMGs with hidden variables

Theorem 1 (Validity) If and only if a valid backdoor adjustment set exists, then **O** is a valid adjustment set.

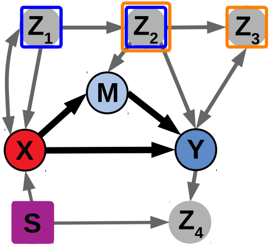




Theorem 2 (O-set vs Adjust set)

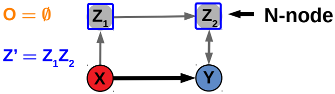
$J_{\mathbf{O}} \geq J_{\mathbf{vancs}}$ for any graph $\mathcal{G}$ (...).

$\Longrightarrow Var(\widehat{\Delta}_{yxx'|\mathbf{s}.\mathbf{o}}) \leq Var(\widehat{\Delta}_{yxx'|\mathbf{s}.\mathbf{adjust}})$



Necessary and sufficient conditions for graphical optimality

Theorem 3 If and only if (...)
(I) for *all* $N \in \mathbf{N} = sp(Y\mathbf{MC}) \setminus (\mathbf{forbOS})$ and all its collider paths i to $W \in Y\mathbf{M}$ (...) it holds that $\mathbf{O}_{\pi_i^N} = \mathbf{O}(X, Y, \mathbf{S}' = \mathbf{S}N\pi_i^N)$ is non-valid, and
(II) for *all* $E \in \mathbf{O} \setminus \mathbf{P}$ with $E \not\perp\!\!\!\perp X \mid \mathbf{SO} \setminus \{E\}$ there exists $E {\leftrightarrow} W$ or $E {*\!\!\rightarrow} C {\leftrightarrow} \cdots {\leftrightarrow} W$ where all colliders $C \in \mathbf{vancs}$,
then $\mathbf{O}$ is optimal for all probability densities consistent with $\mathcal{G}$.



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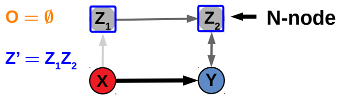

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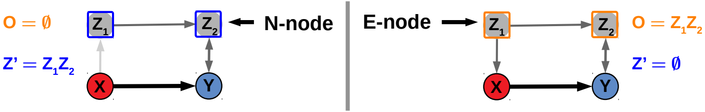

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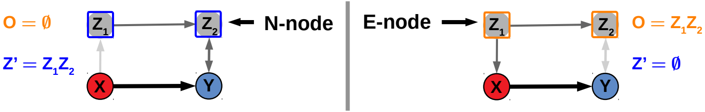
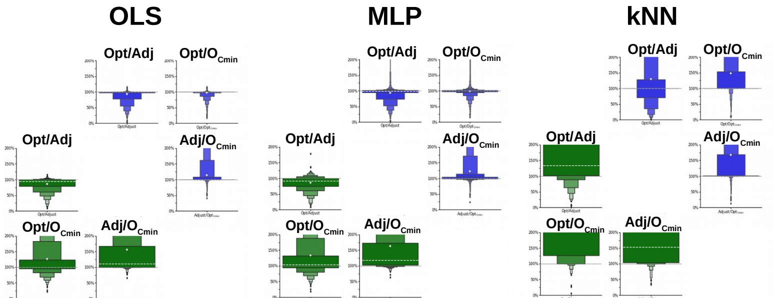


Numerical experiments in paper

- Among 12,000 randomly created configurations **95%** fulfill optimality!



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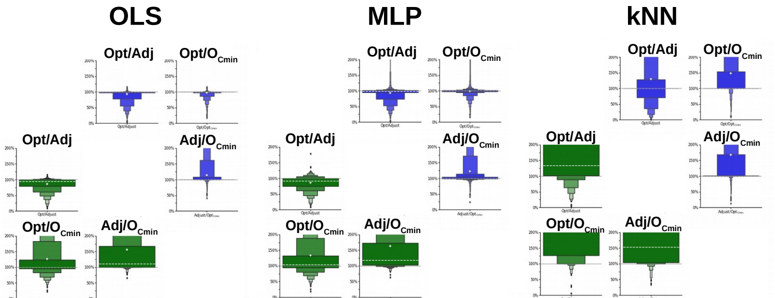

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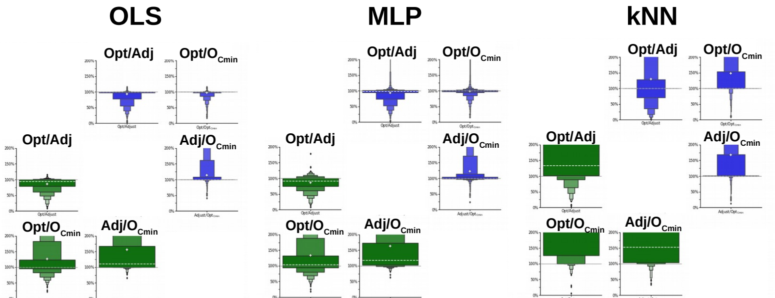

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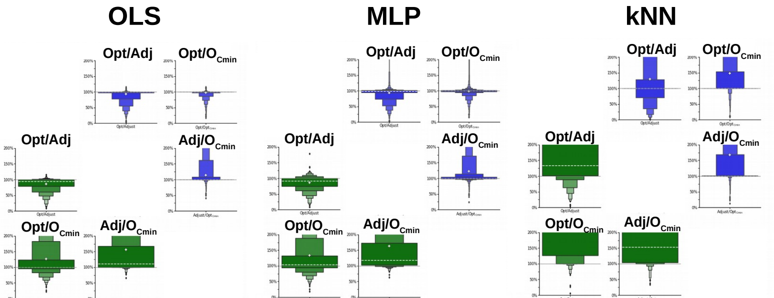
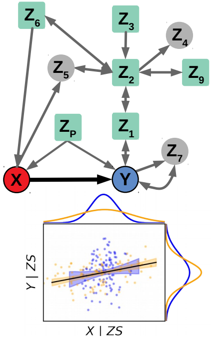
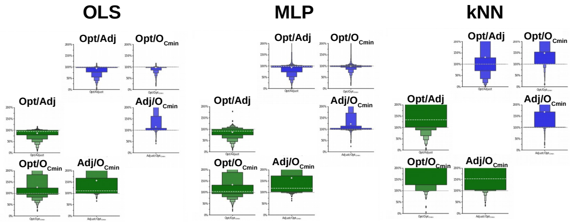


Summary

- Theorem 3 completely characterizes graphical optimality for ADMGs (and DMAGs)





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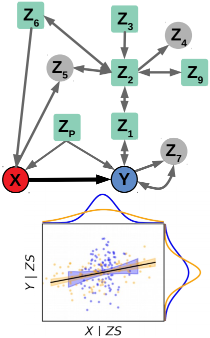
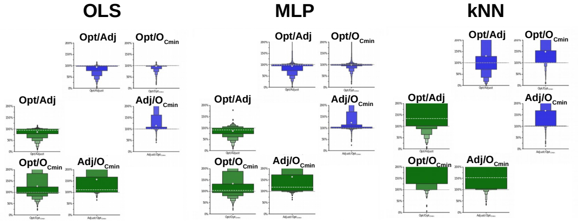
  $\rightarrow$ natural choice in automated causal inference





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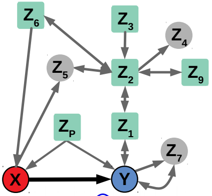
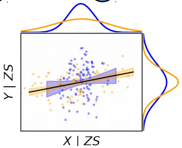
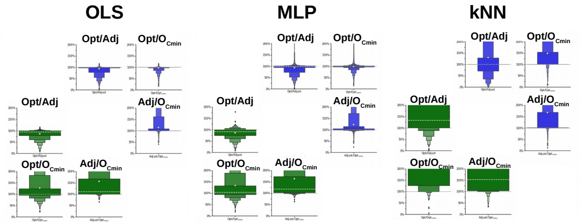
- Python code: `https://github.com/jakobrunge/tigramite`







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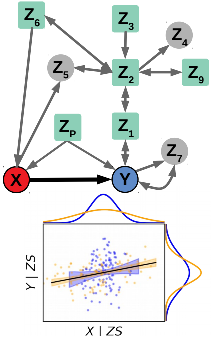
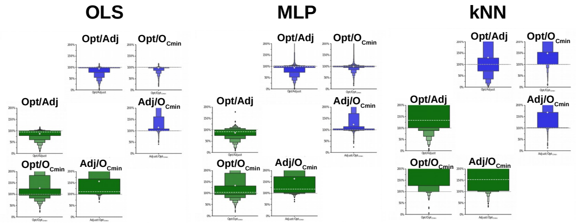
- Python code: `https://github.com/jakobrunge/tigramite`
- Open questions: Theory for non-parametric estimators, PAGs, ...





Thank you! Questions?

- *Nature Comm.* Perspective on causal discovery in time series [Runge et al., 2019a]
- Causal inference: full theory [Pearl, 2009], primer [Pearl et al., 2016], linear models [Pearl, 2013], popular science book [Pearl and Mackenzie, 2018]
- Causal discovery: general [Spirtes et al., 2000], for time series [Runge, 2018, Runge et al., 2019a]
- Restricted SCMs [Peters et al., 2017]
- PCMCI [Runge et al., 2019b] in *Science Advances*
- PCMCI$^+$ [Runge, 2020] in *UAI*
- LPCMCI [Gerhardus and Runge, 2020] in *NeurIPS*
- Optimal adjustment [Runge, 2021] in *NeurIPS*
- My software: `jakobrunge.github.io/tigramite`

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