Topological Attention for Time Series Forecasting

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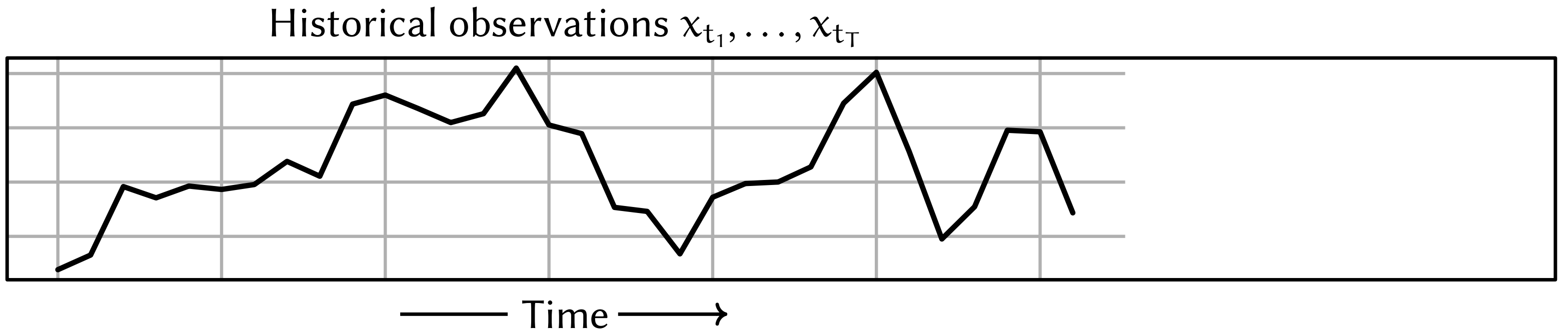
[†]University of Salzburg

NeurIPS 2021

Problem statement

We consider the problem of point-forecasting of **univariate** time series.



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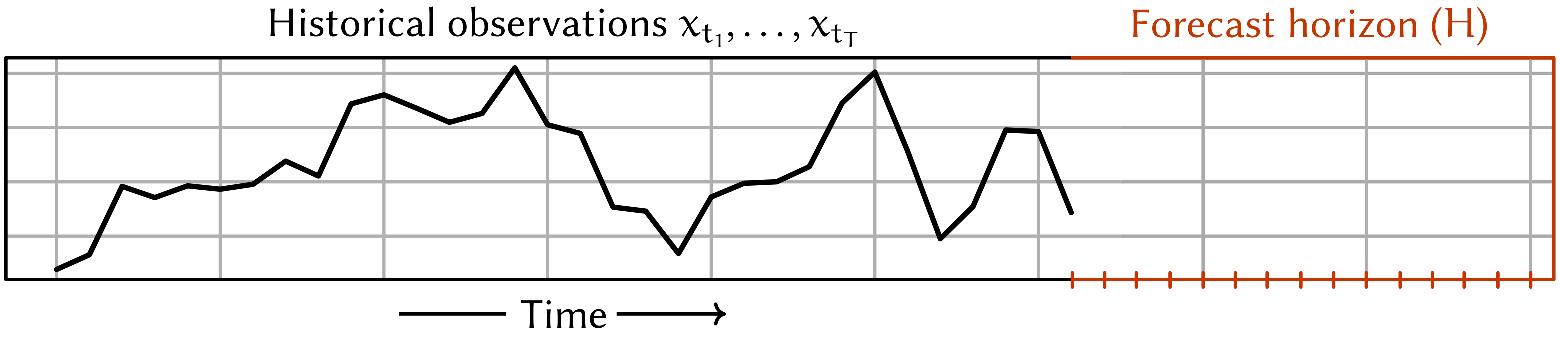

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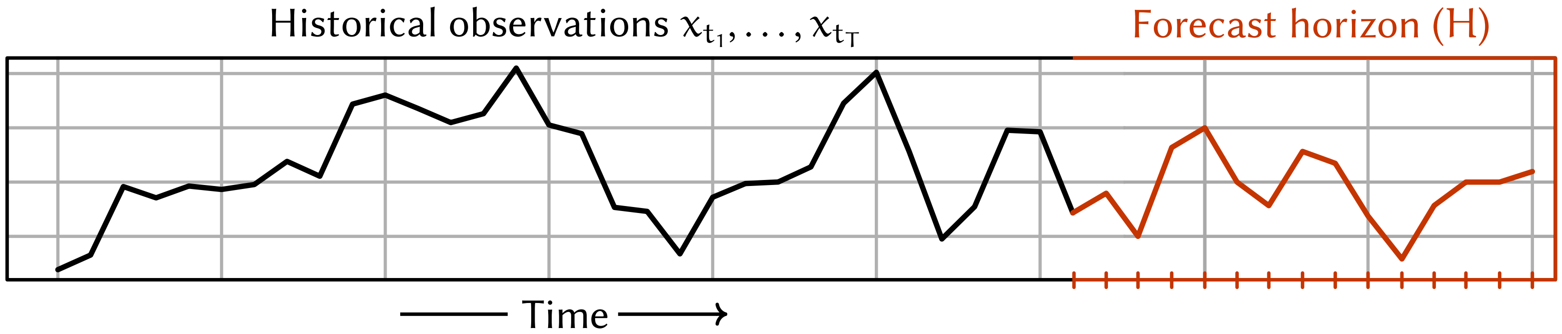

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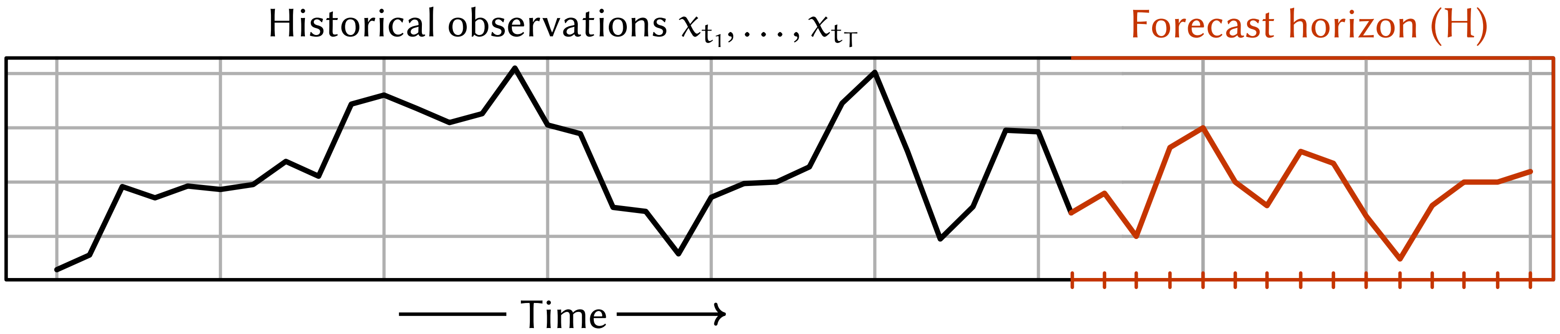

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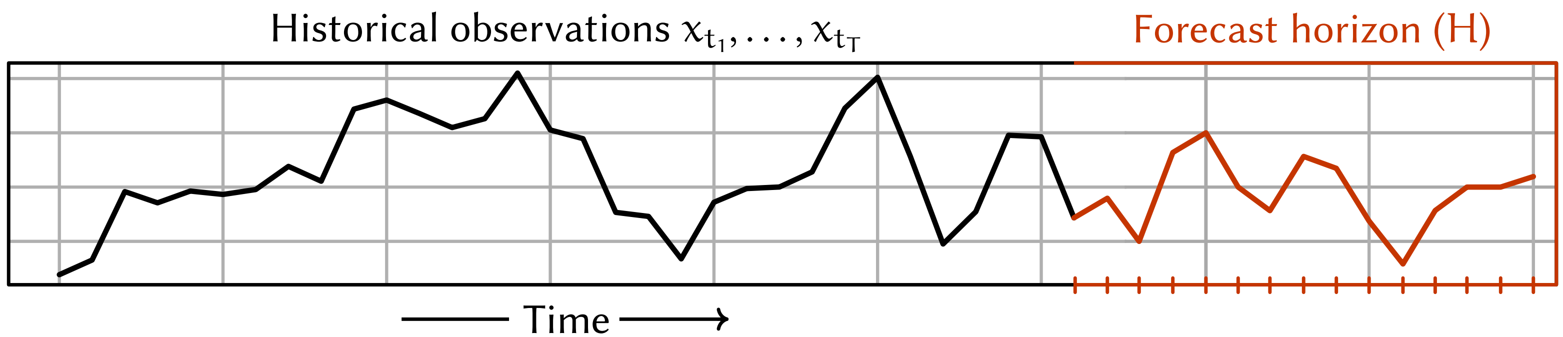

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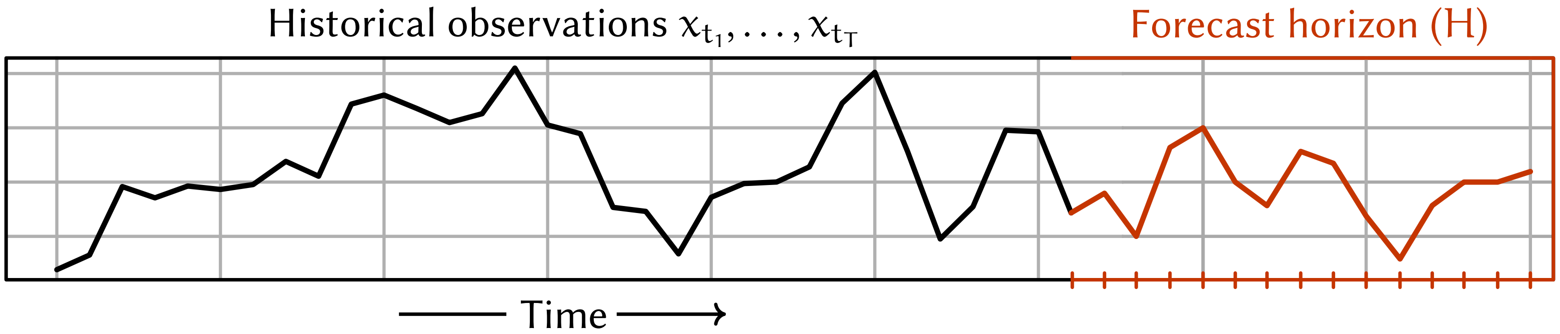

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Q: Can we leverage **topological** information?

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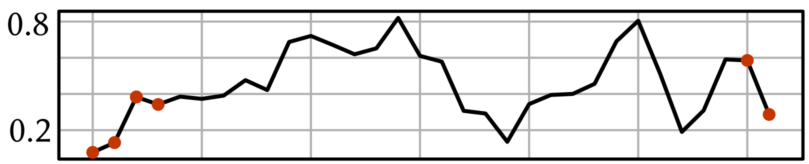
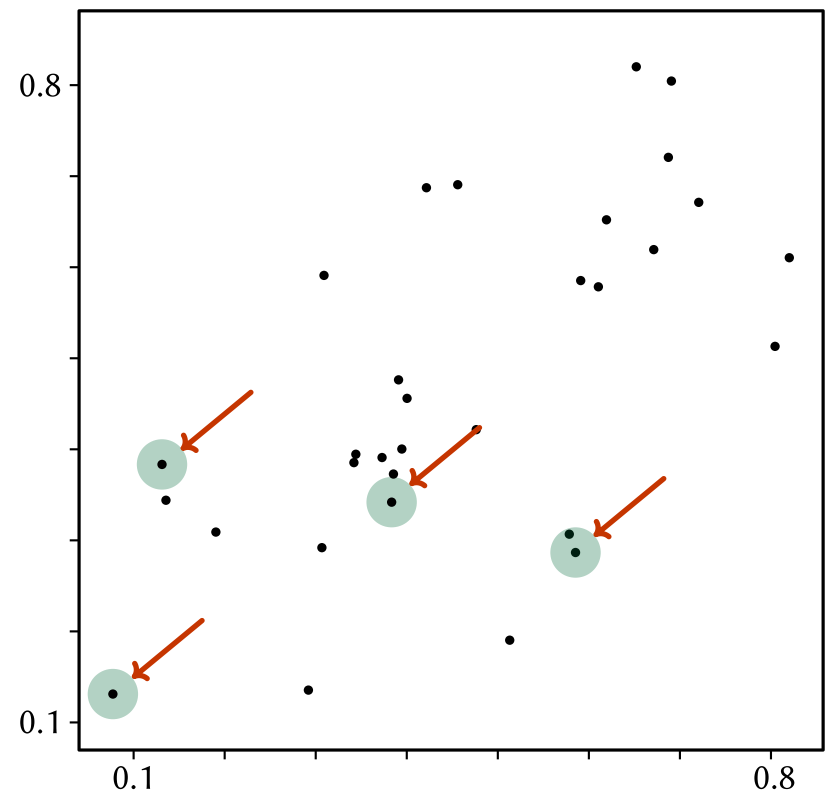
Classicially, one would first embed the time series into $\mathbb{R}^n$ ...

[de Silva et al., 2012; Perea & Harer, 2015]



Example of an embedding in $\mathbb{R}^2$



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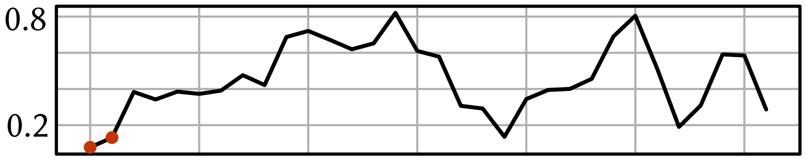

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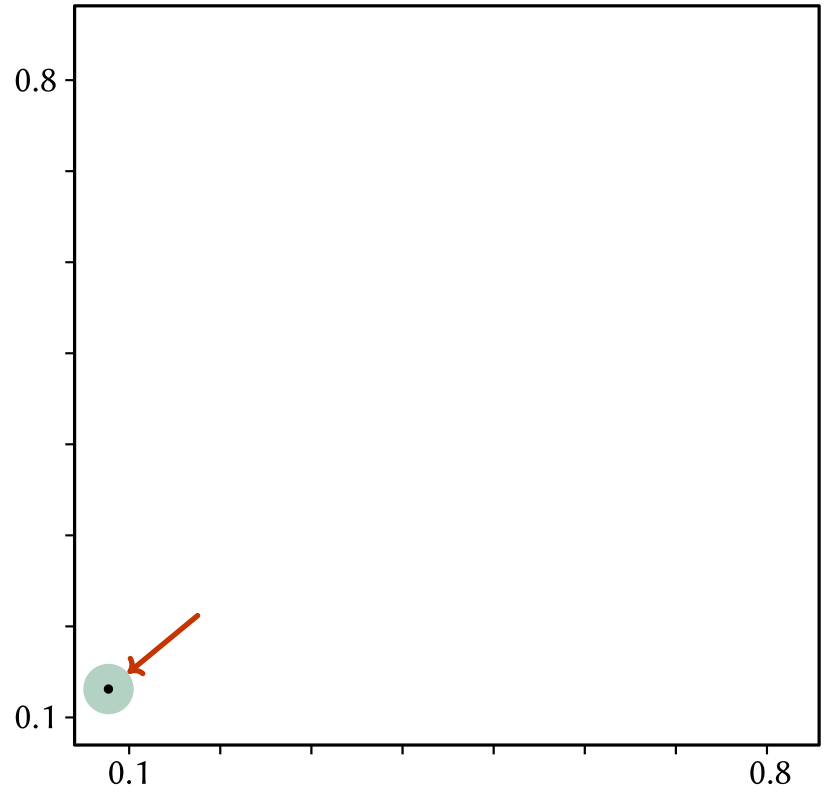

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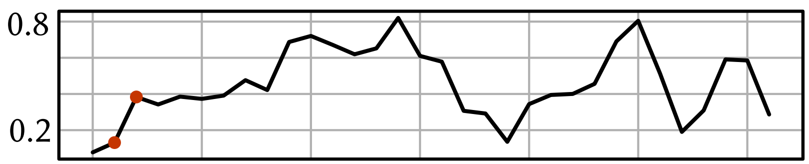

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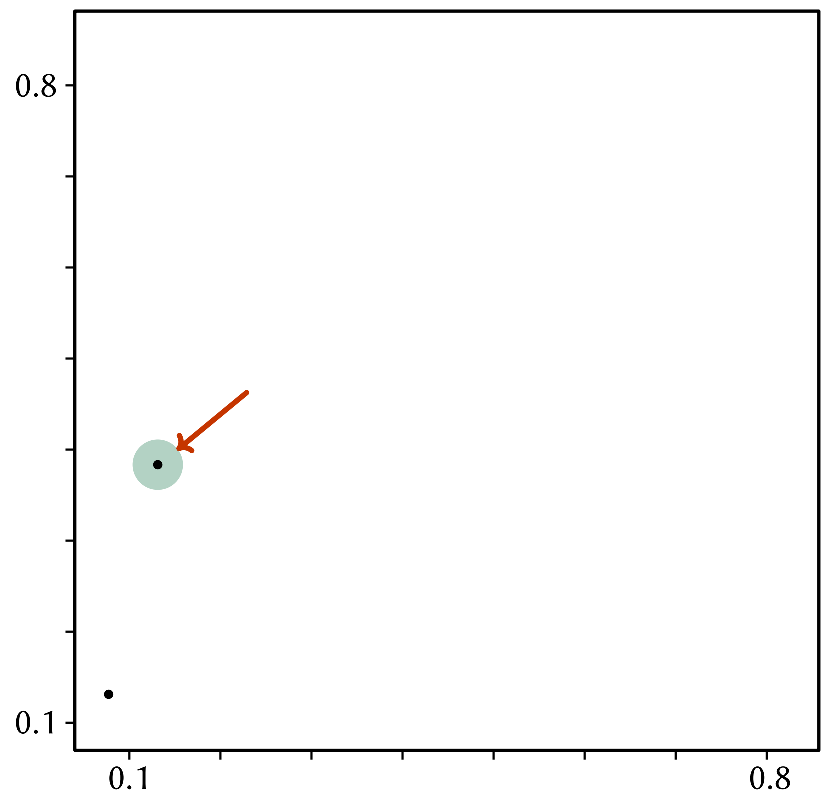

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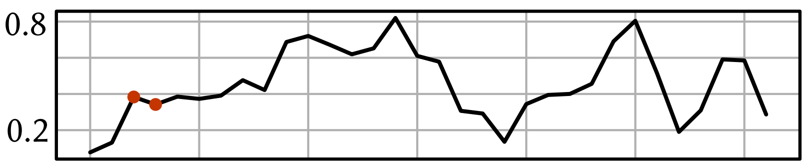

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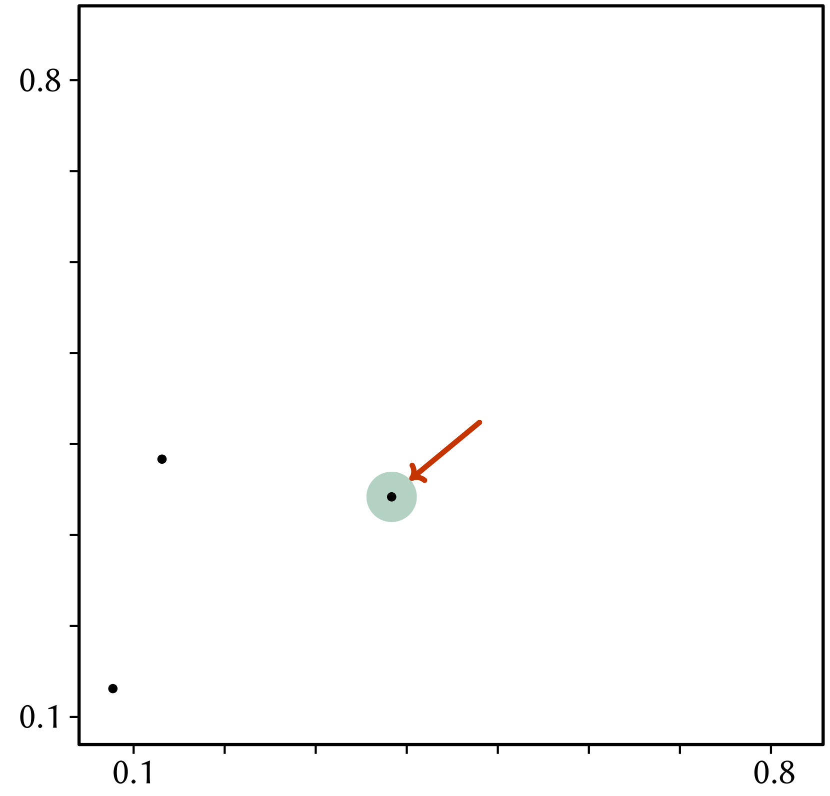

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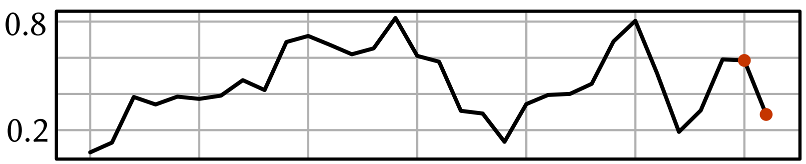

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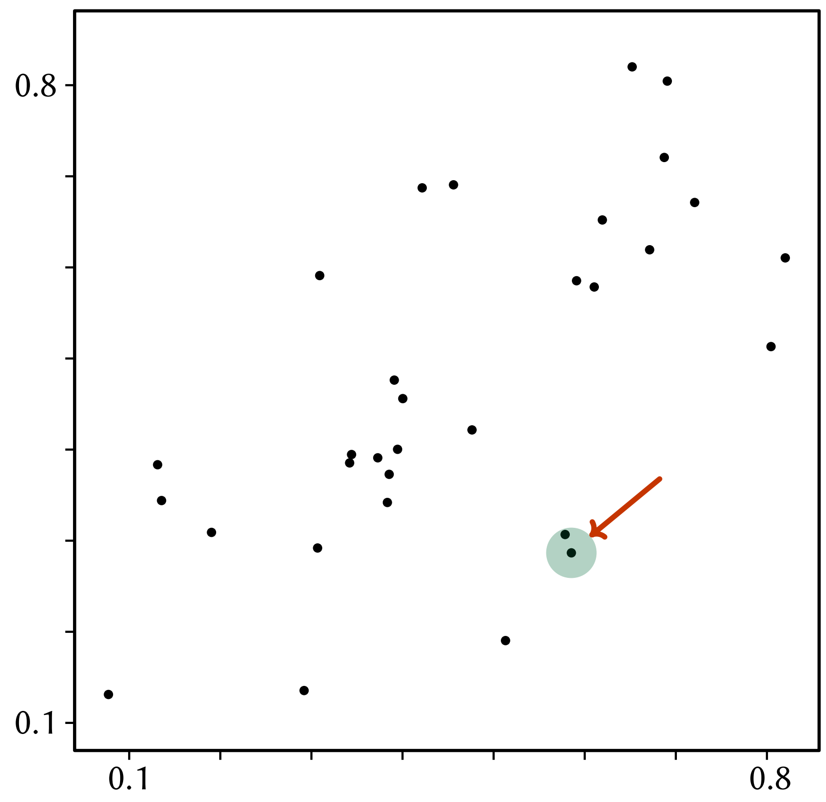

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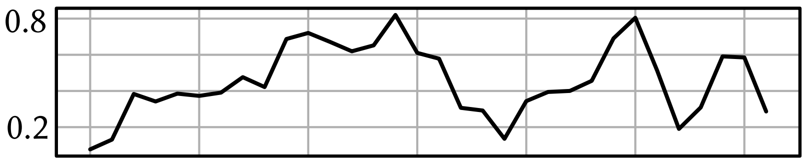

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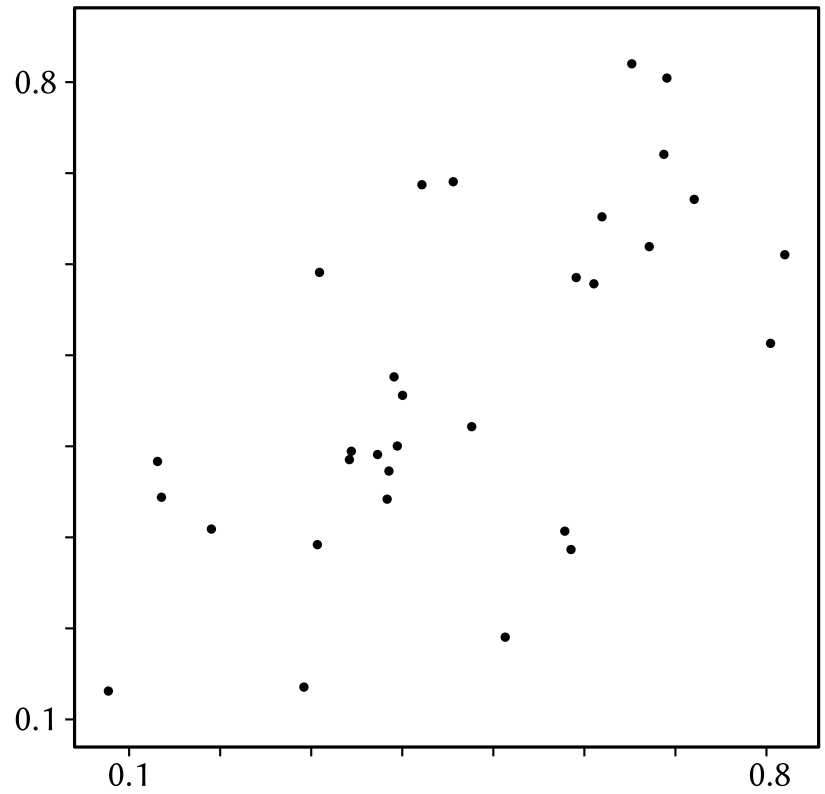

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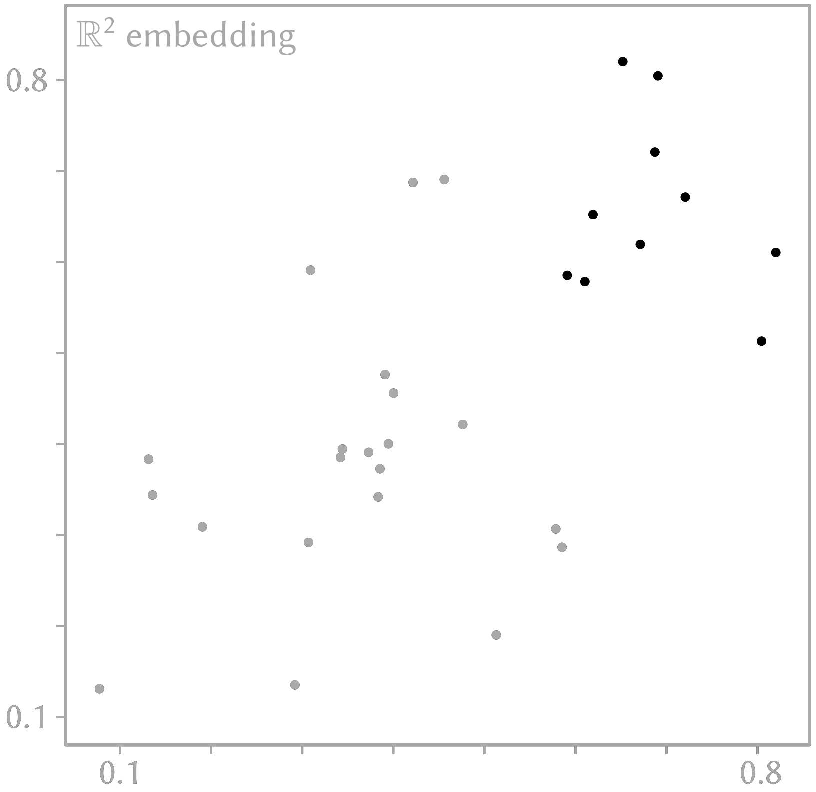

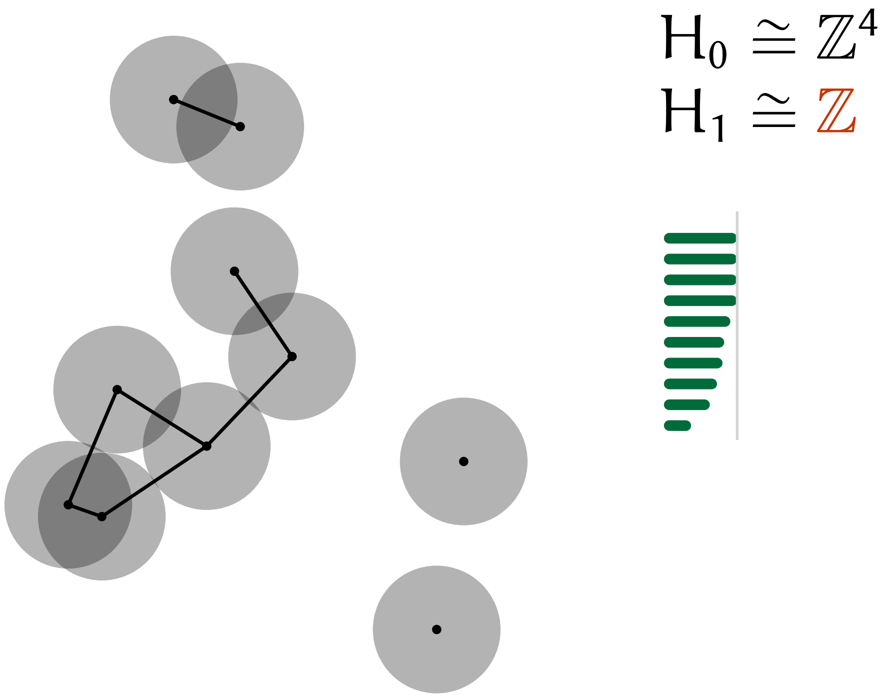

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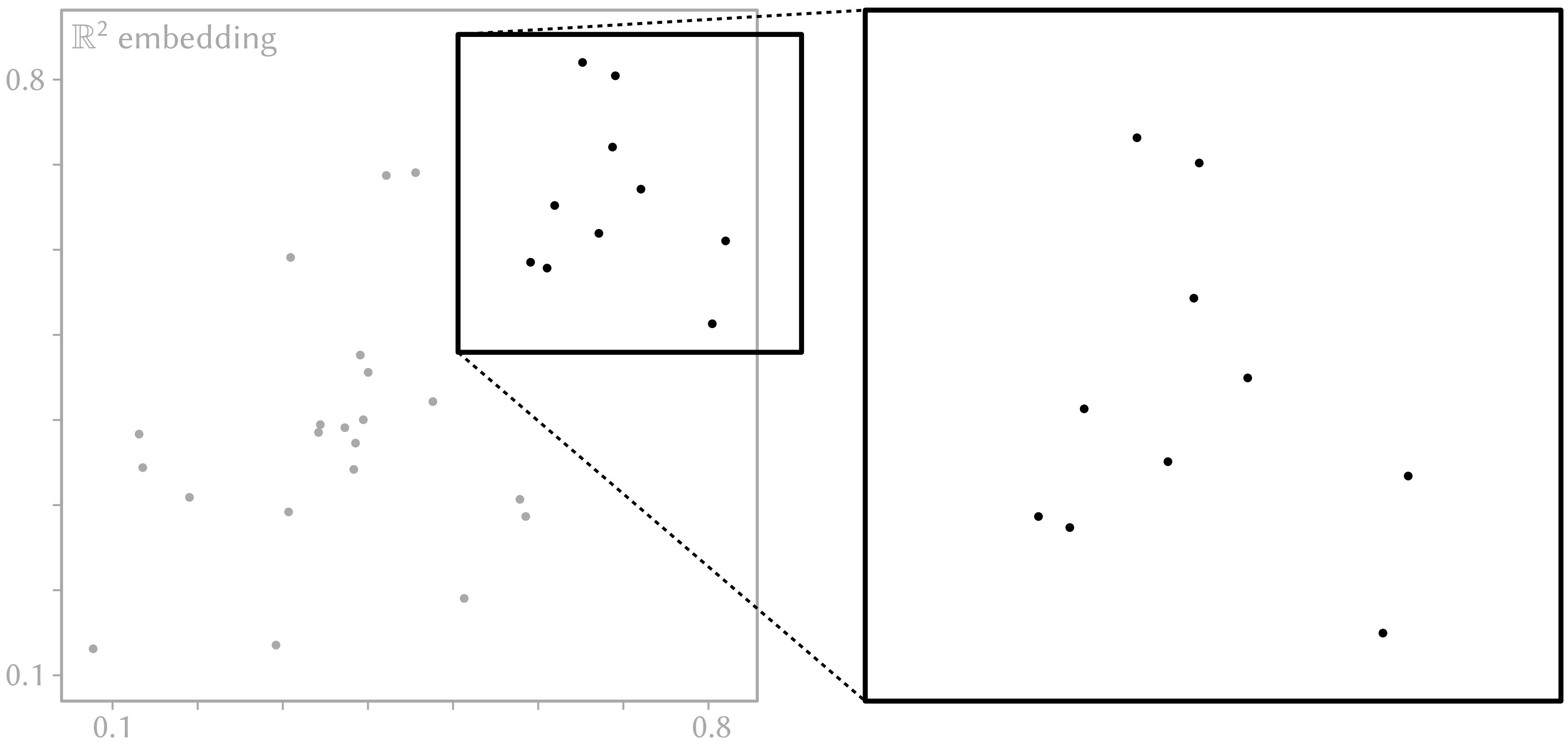

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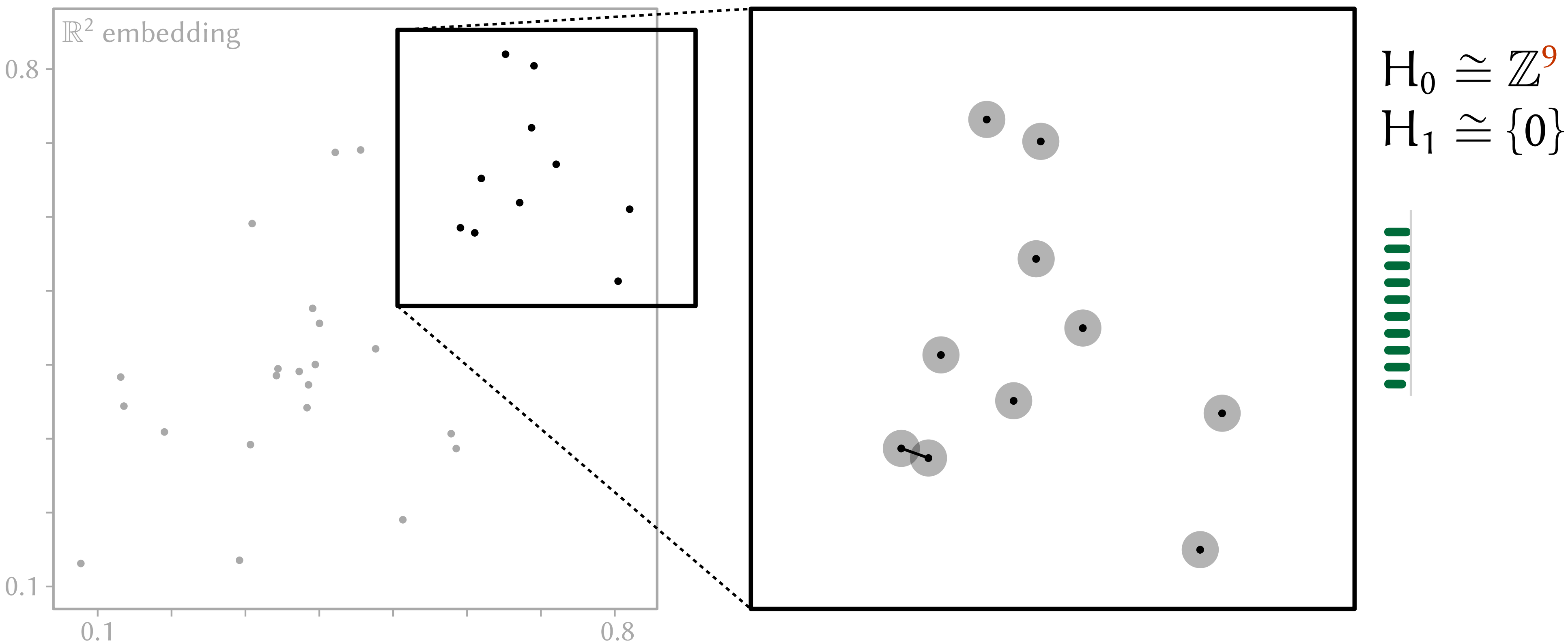

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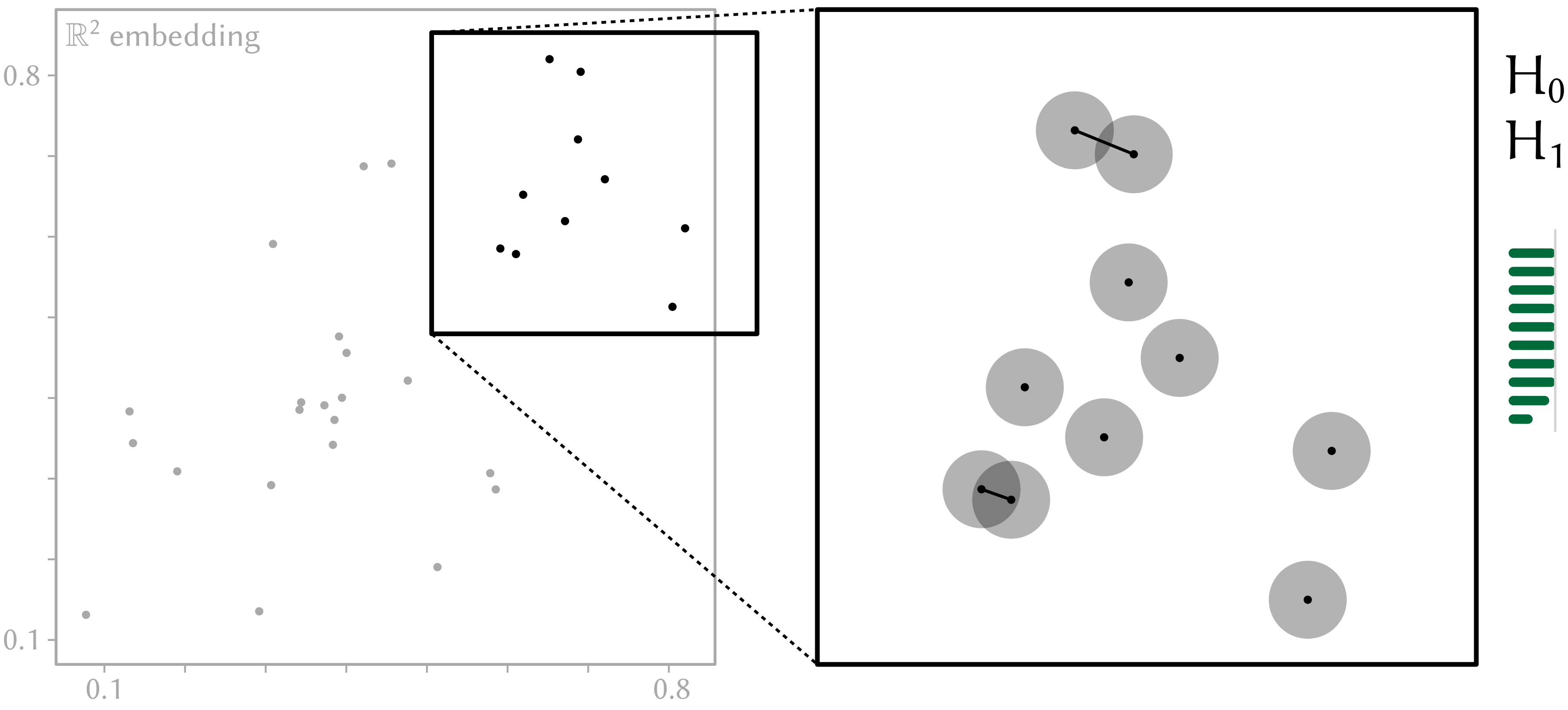

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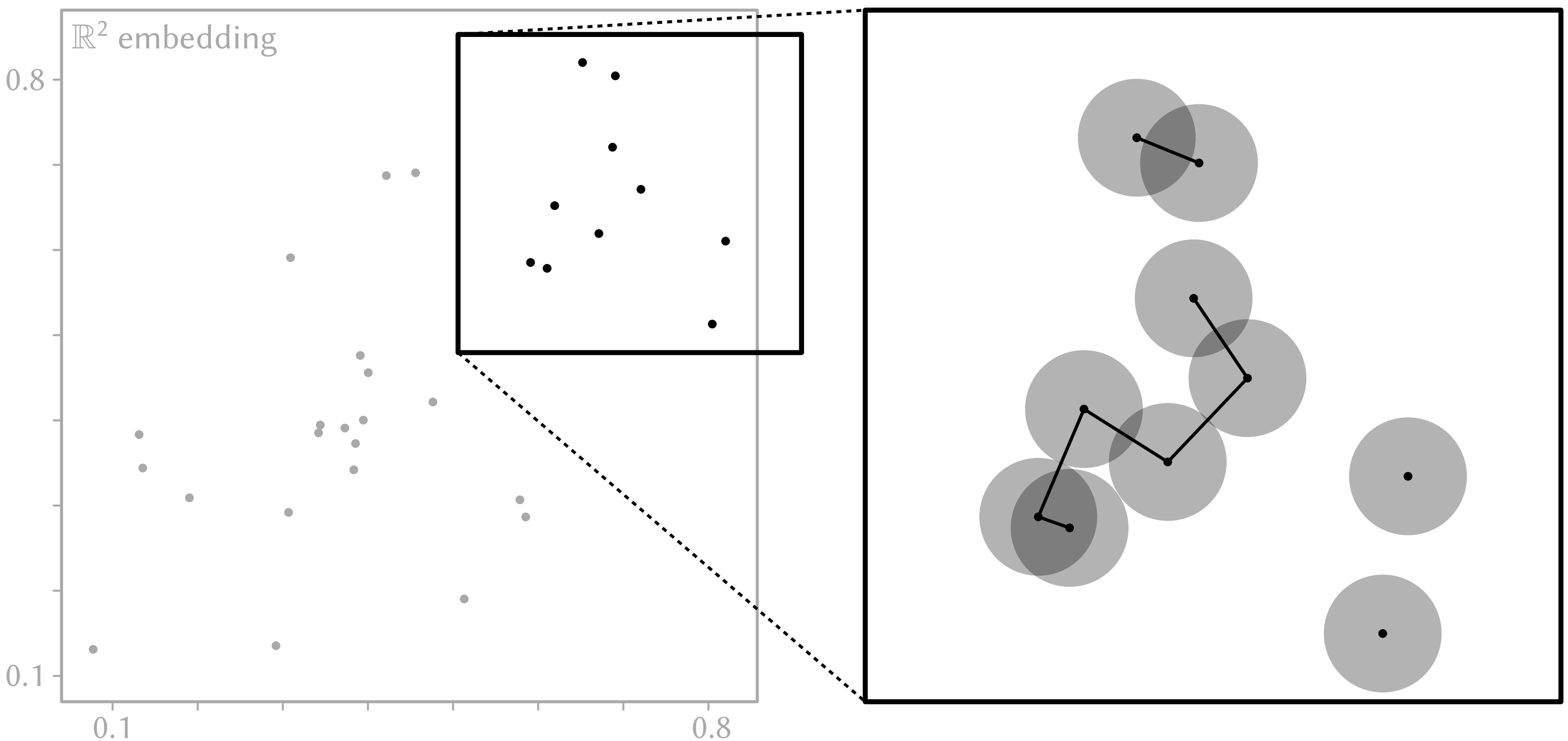

$H_0 \cong \mathbb{Z}^4$
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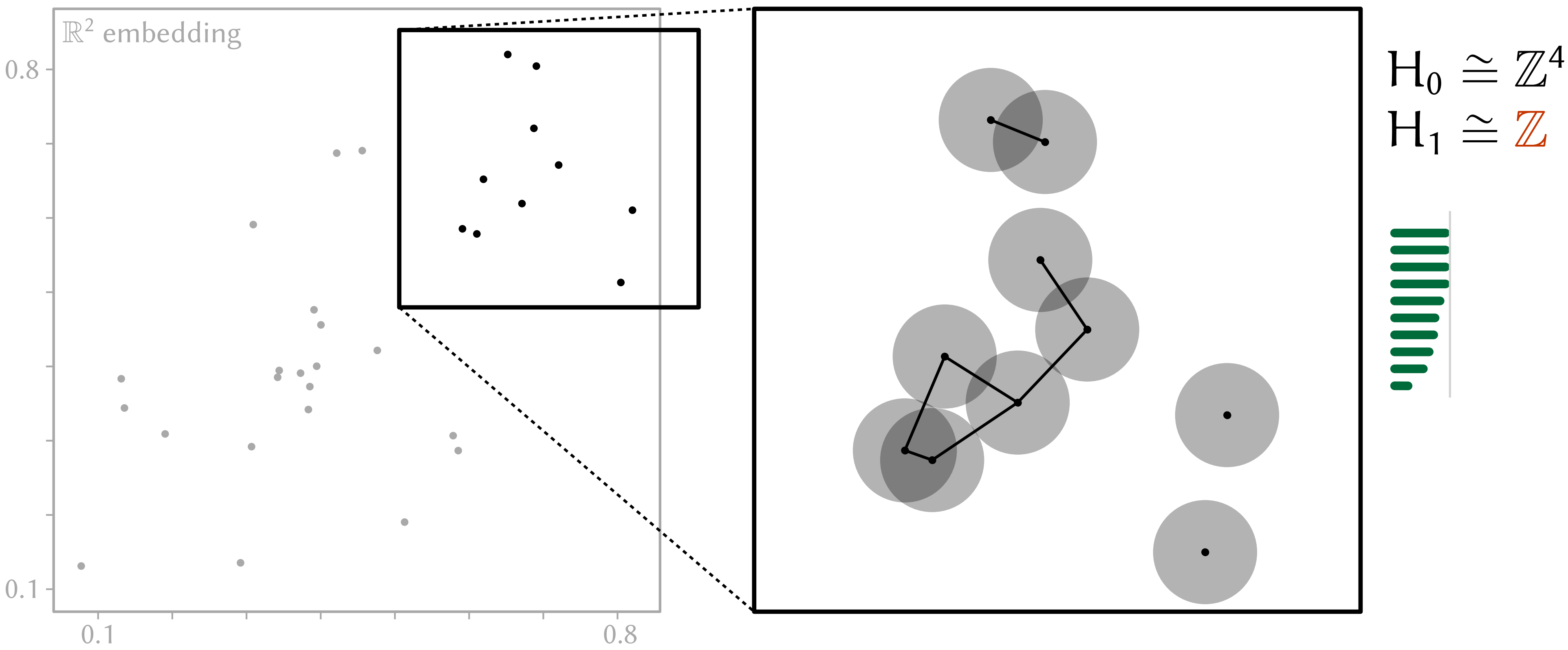

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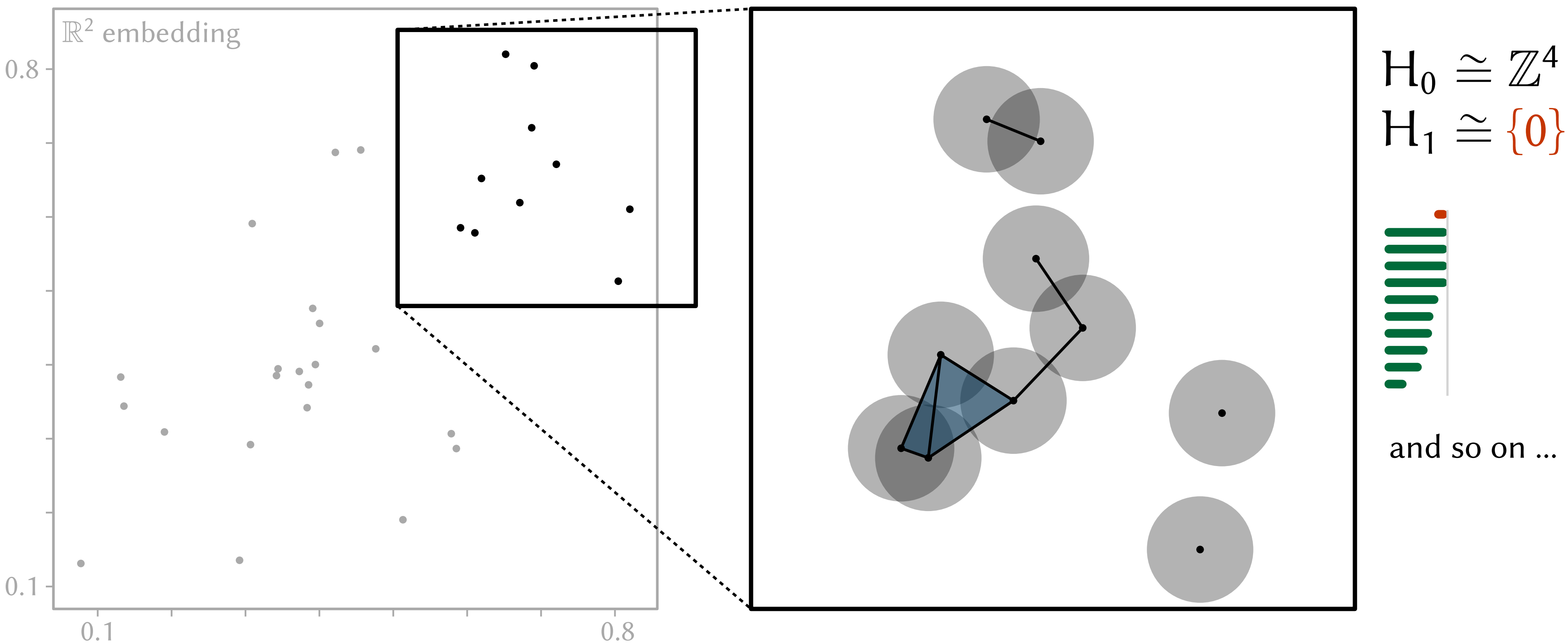

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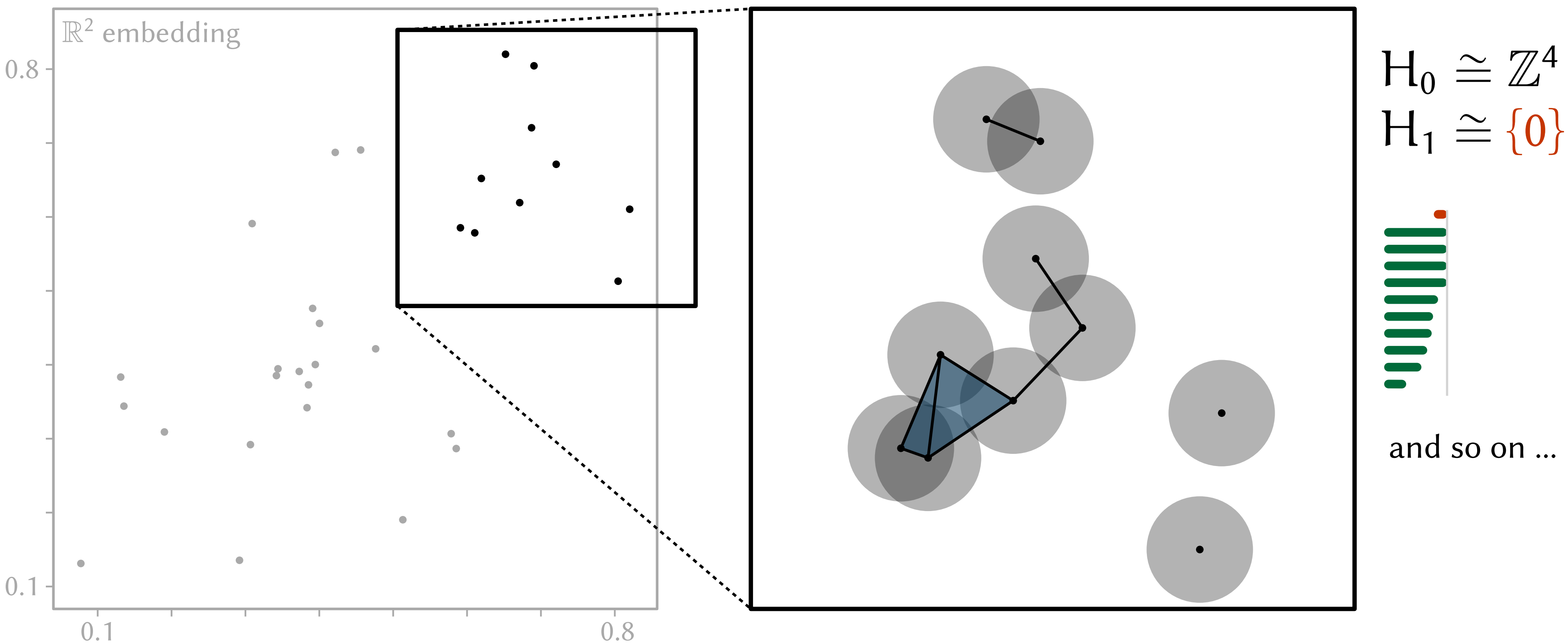

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This tracks **global** topological changes!

(and *locality* is lost)

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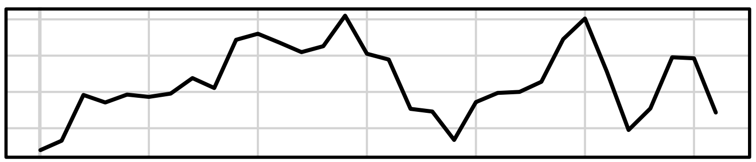
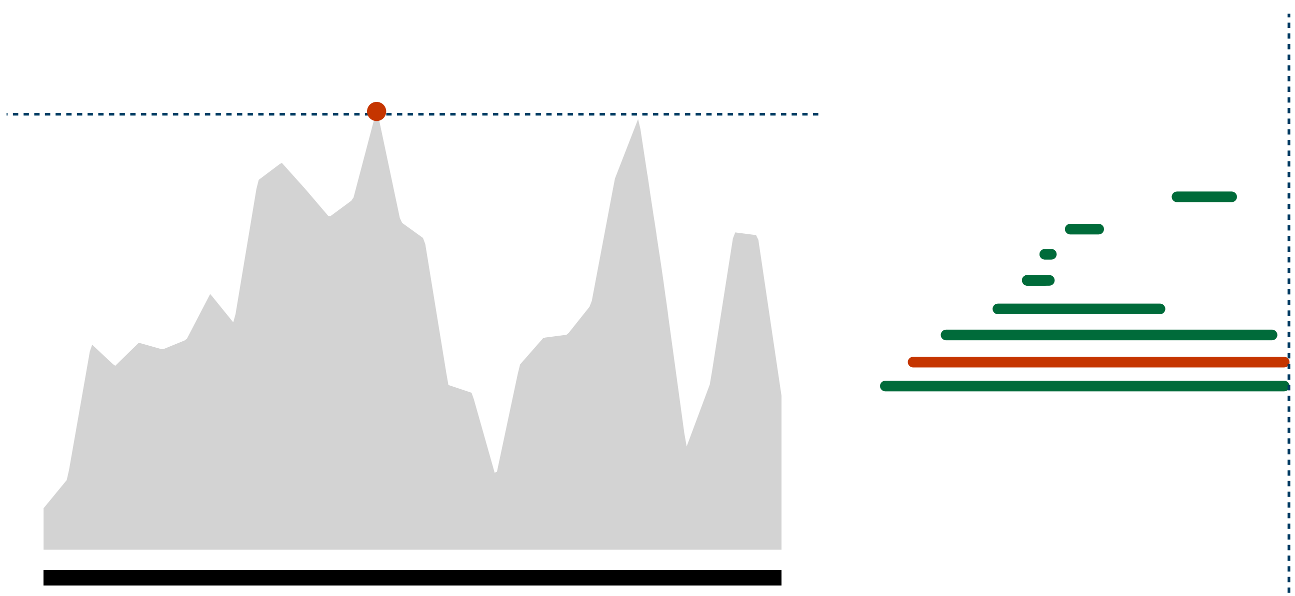
Local topological features for time series

Why not compute such topological features across **sliding windows**?





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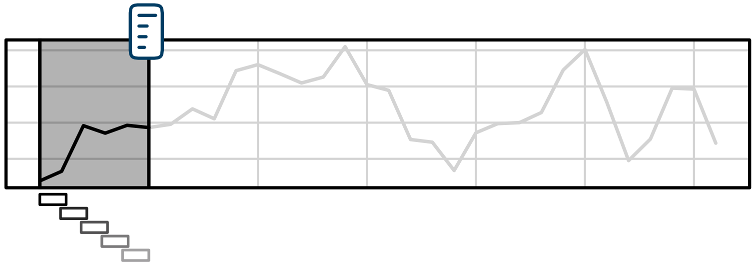

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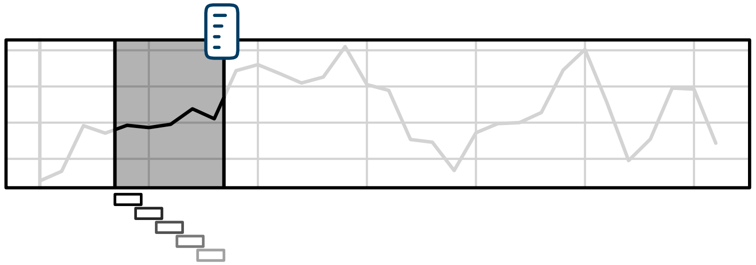

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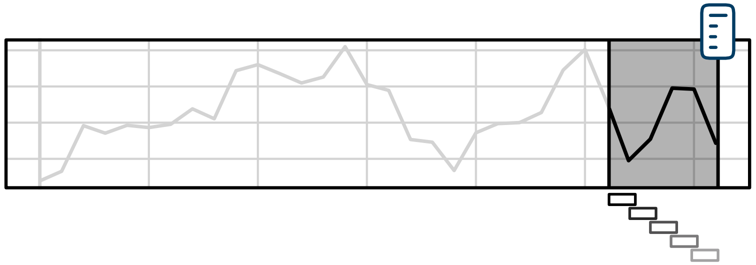

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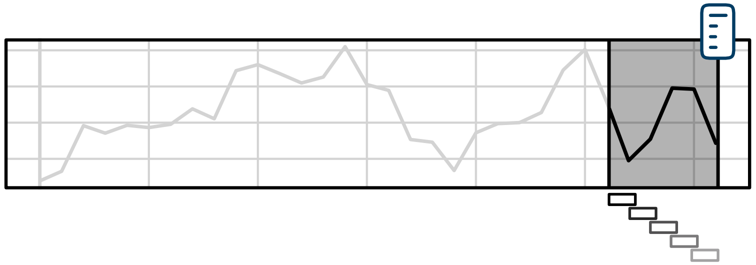

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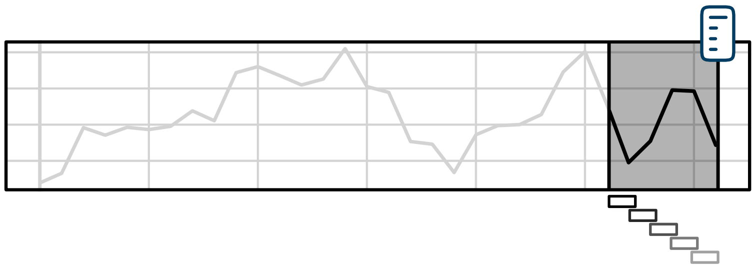

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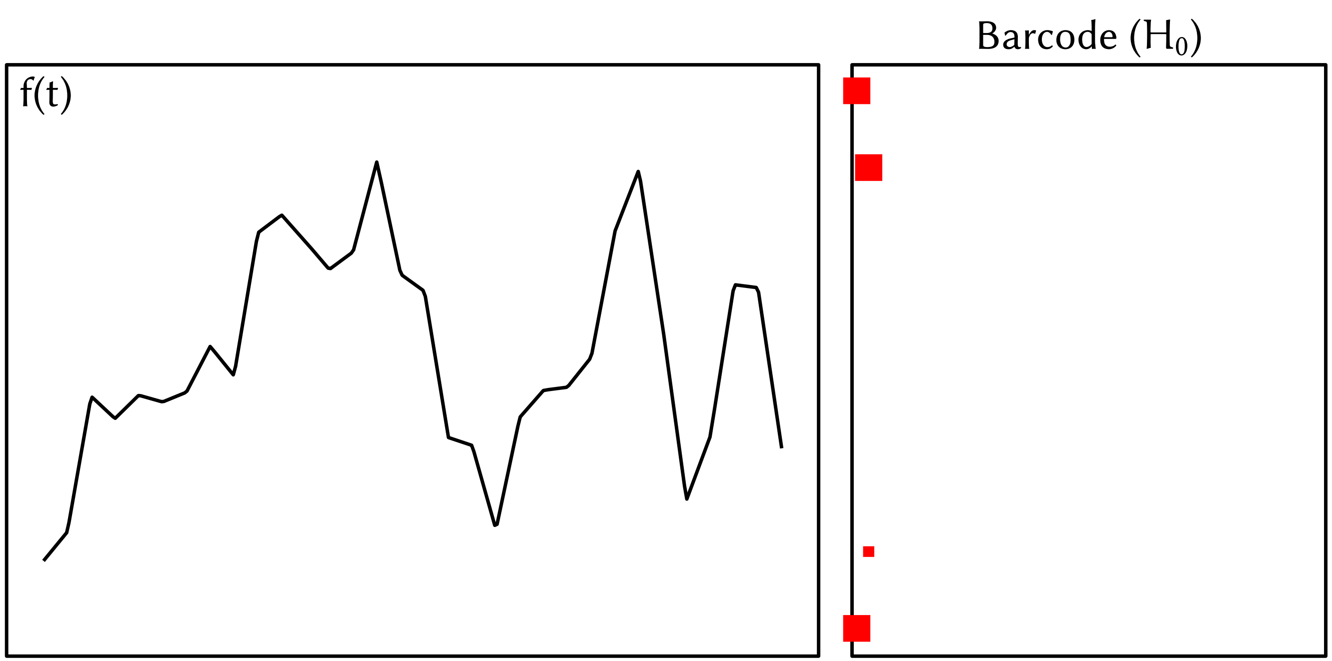

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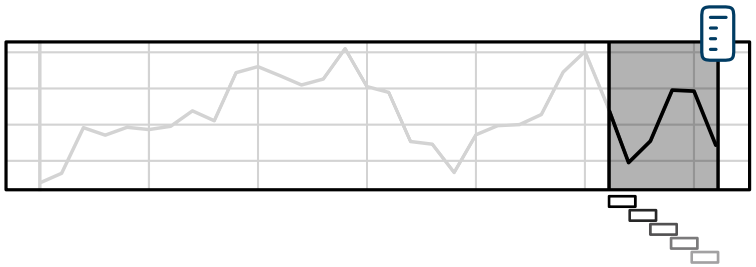

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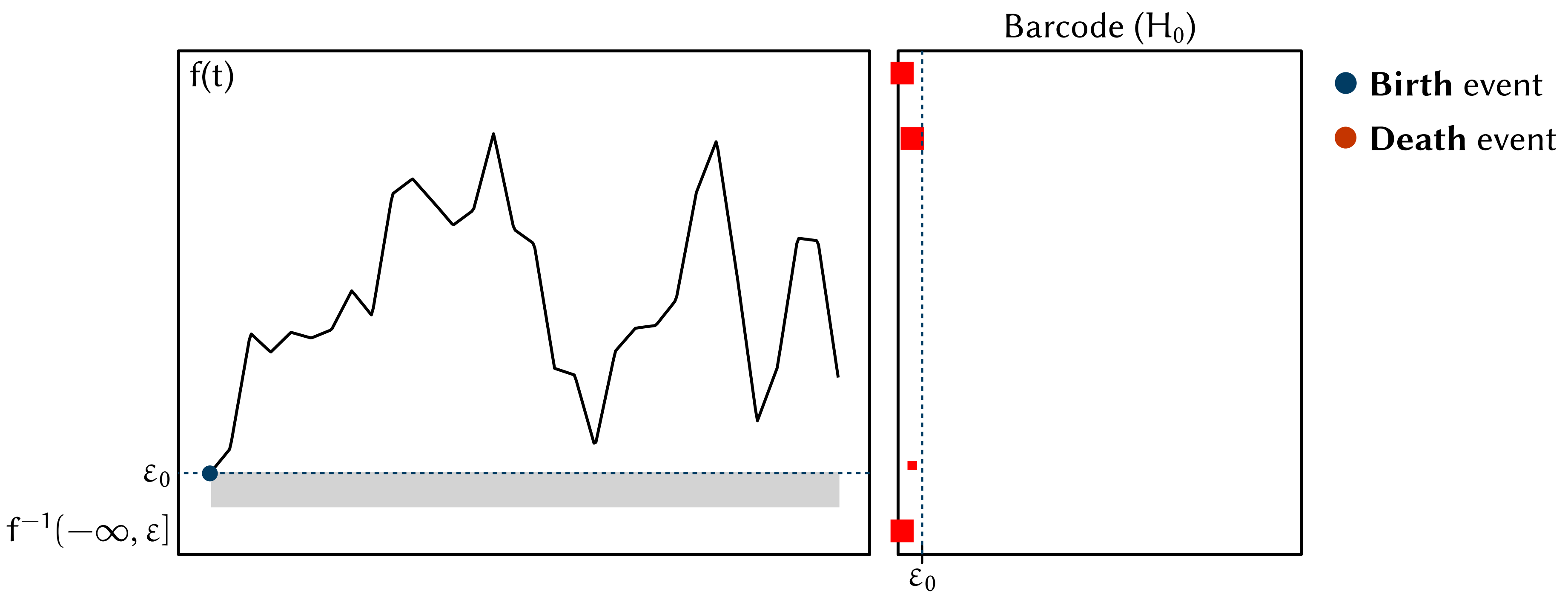

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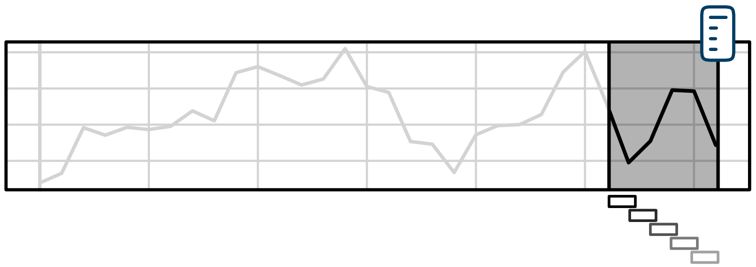

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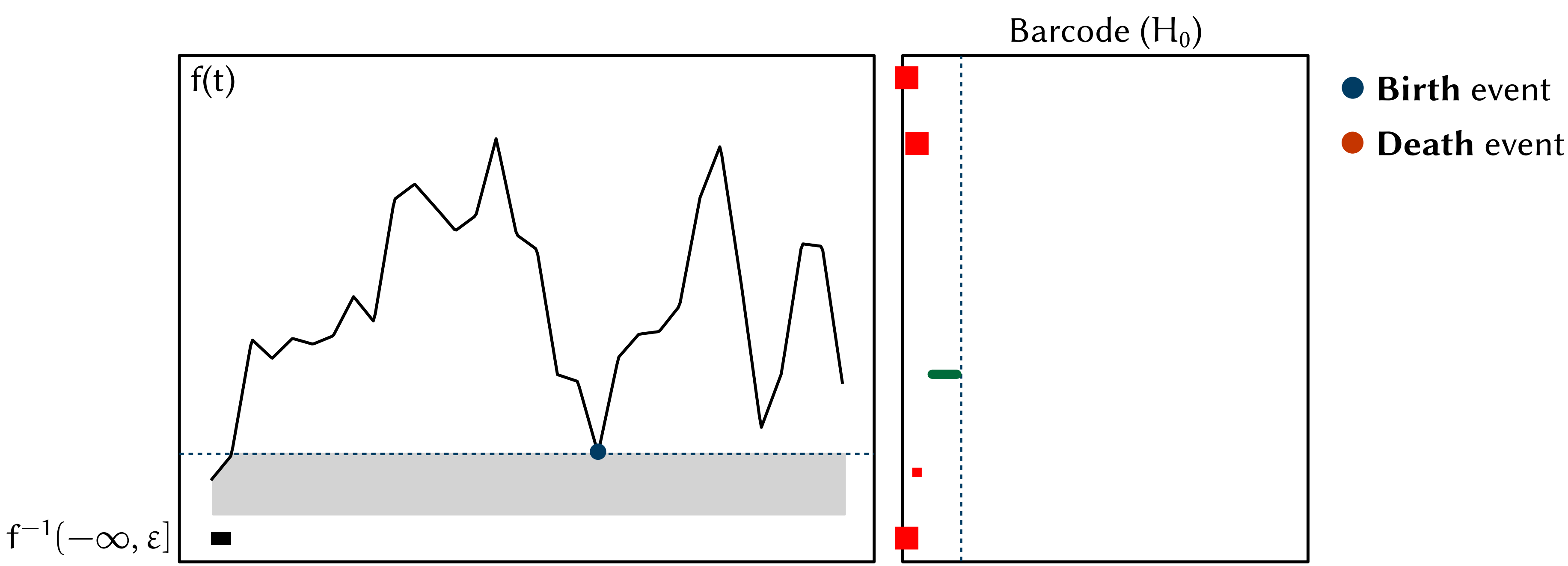

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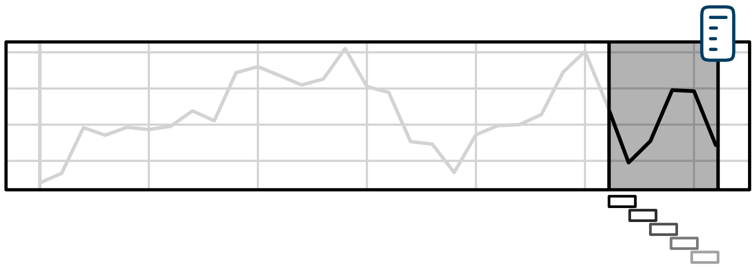

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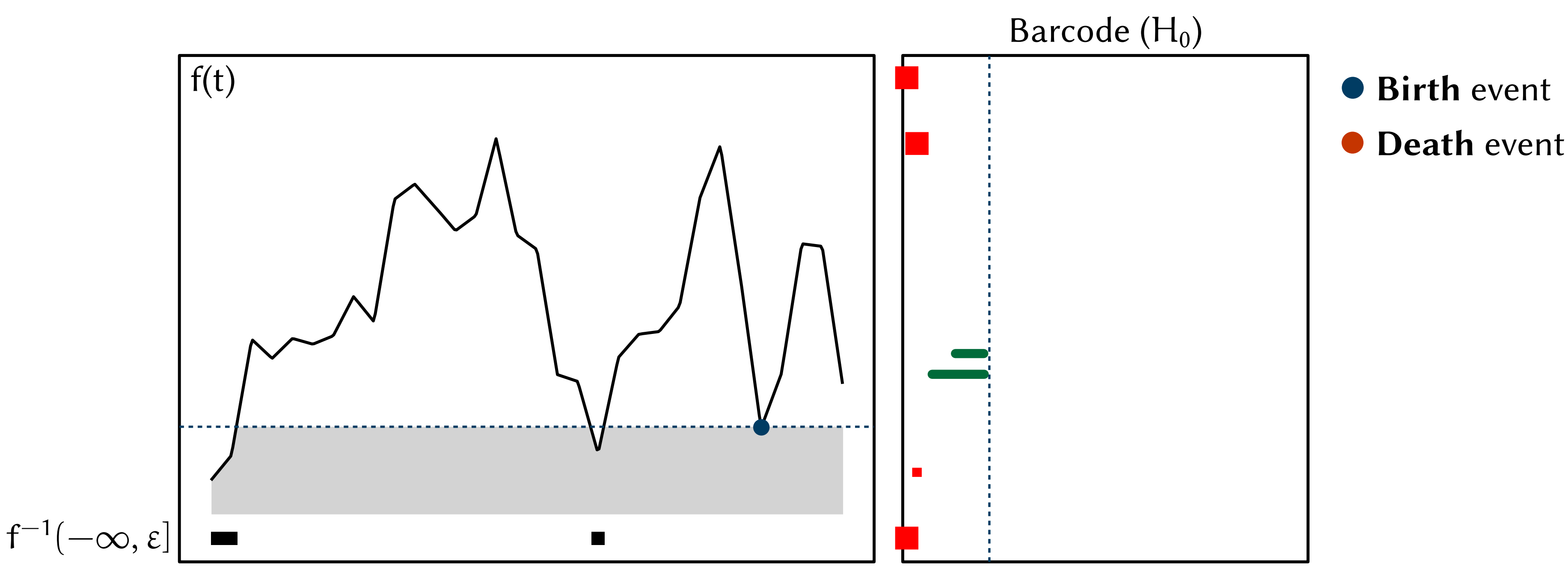

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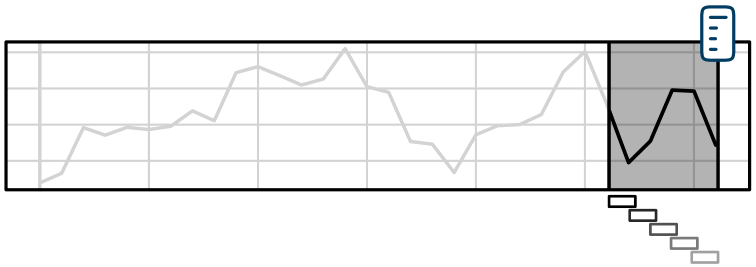

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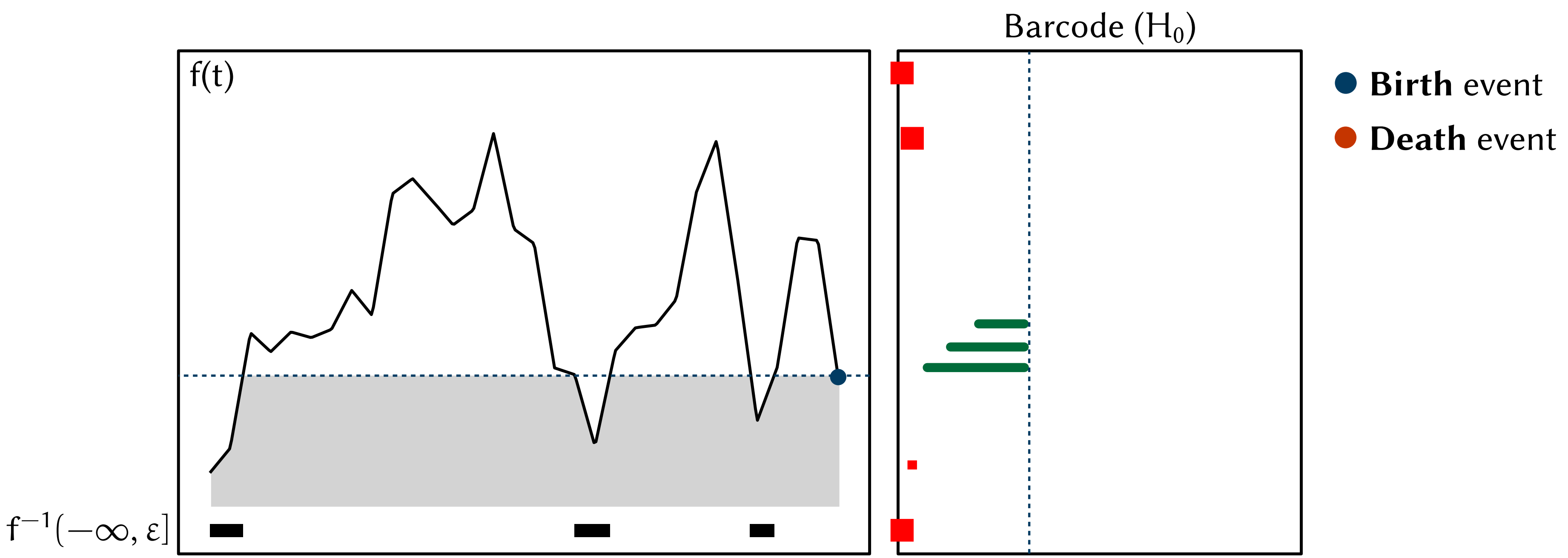

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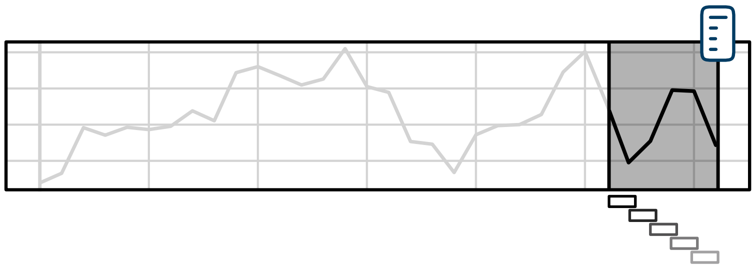

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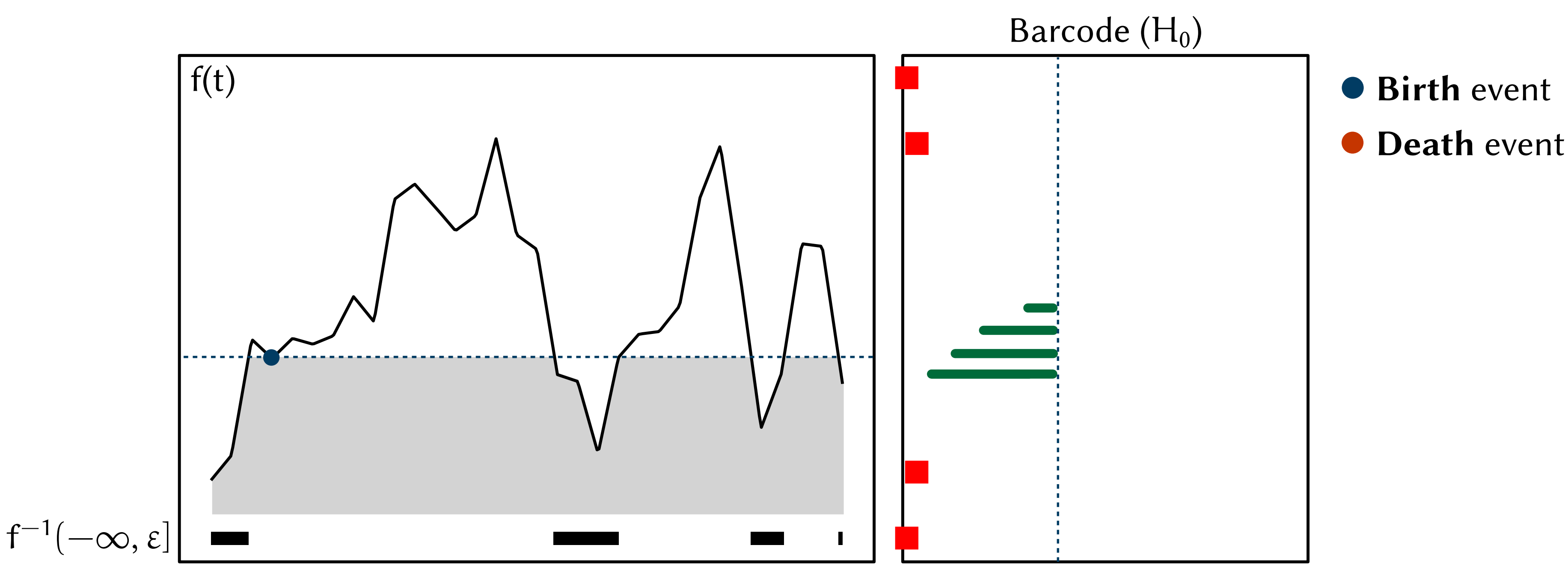

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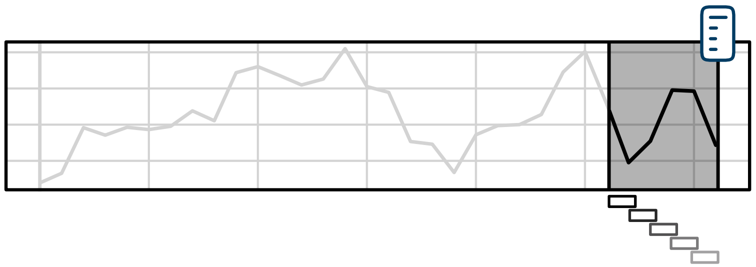

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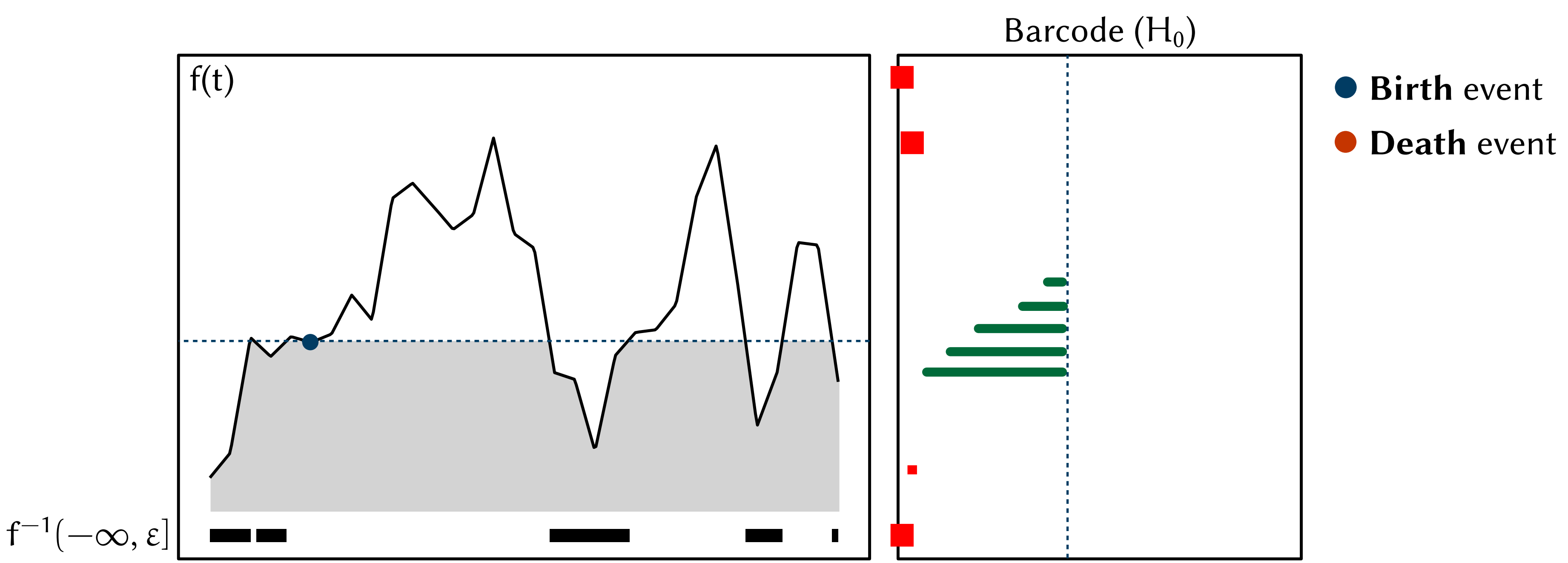

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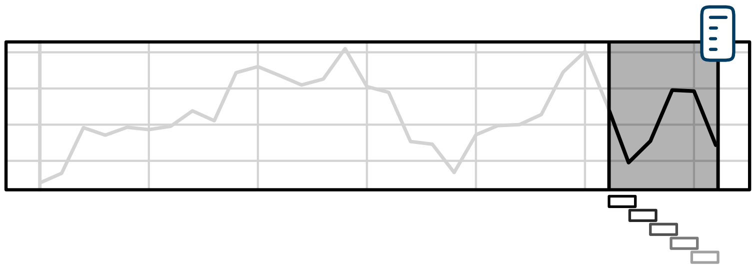

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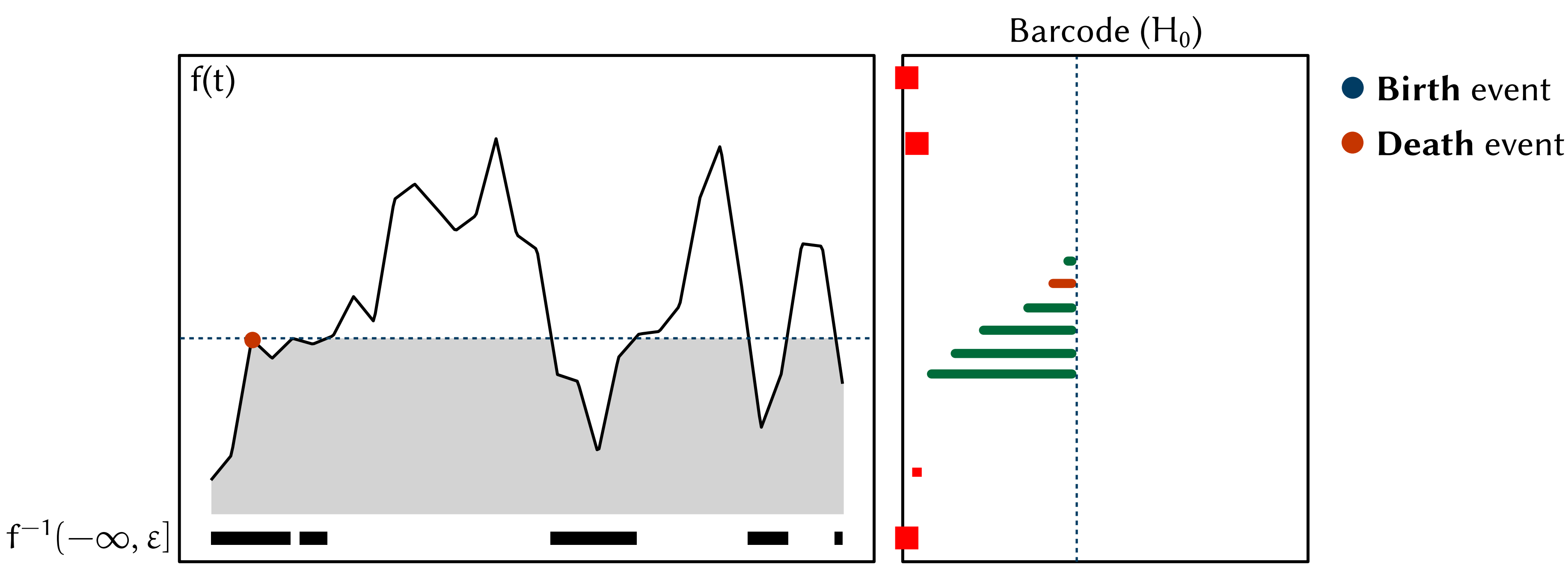

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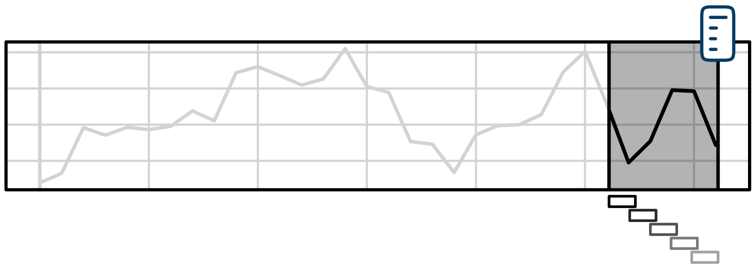

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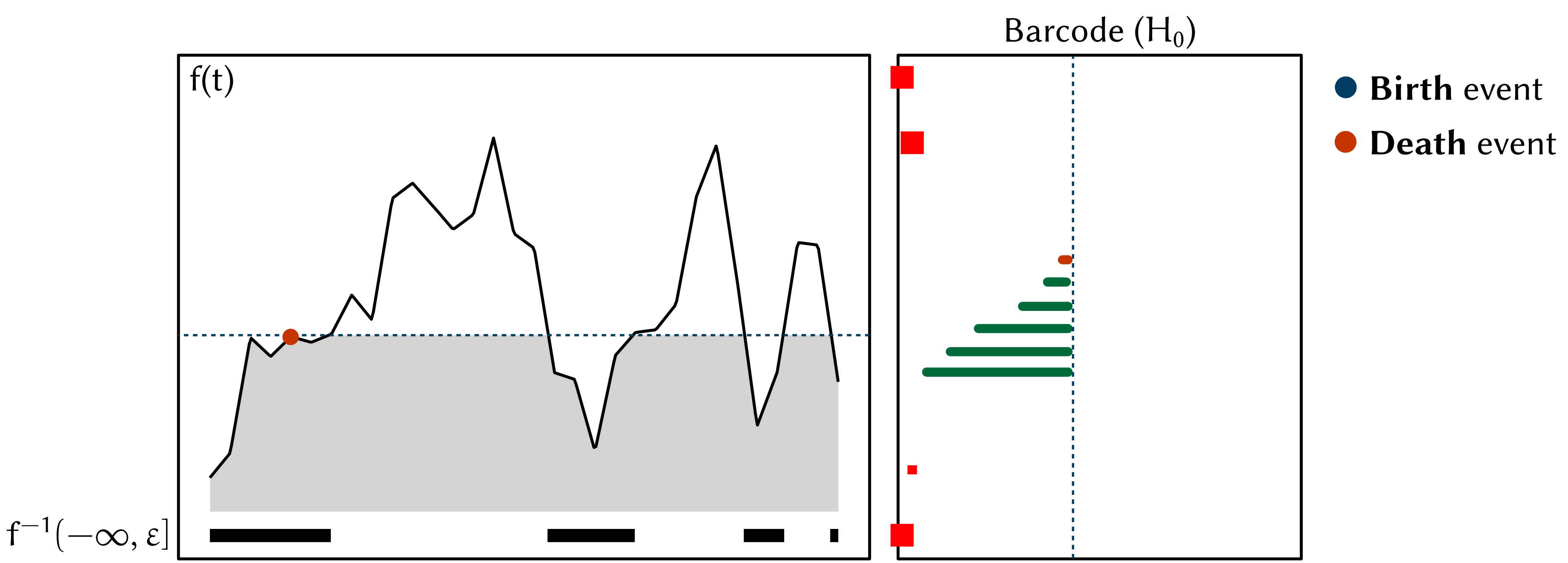

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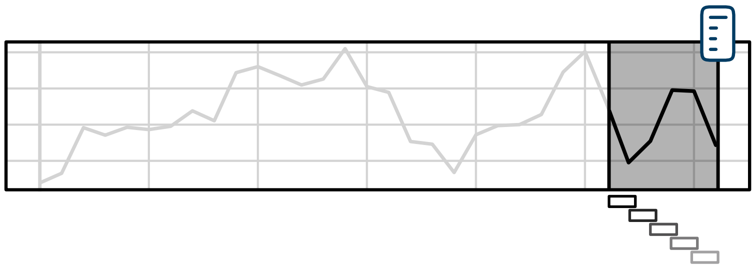

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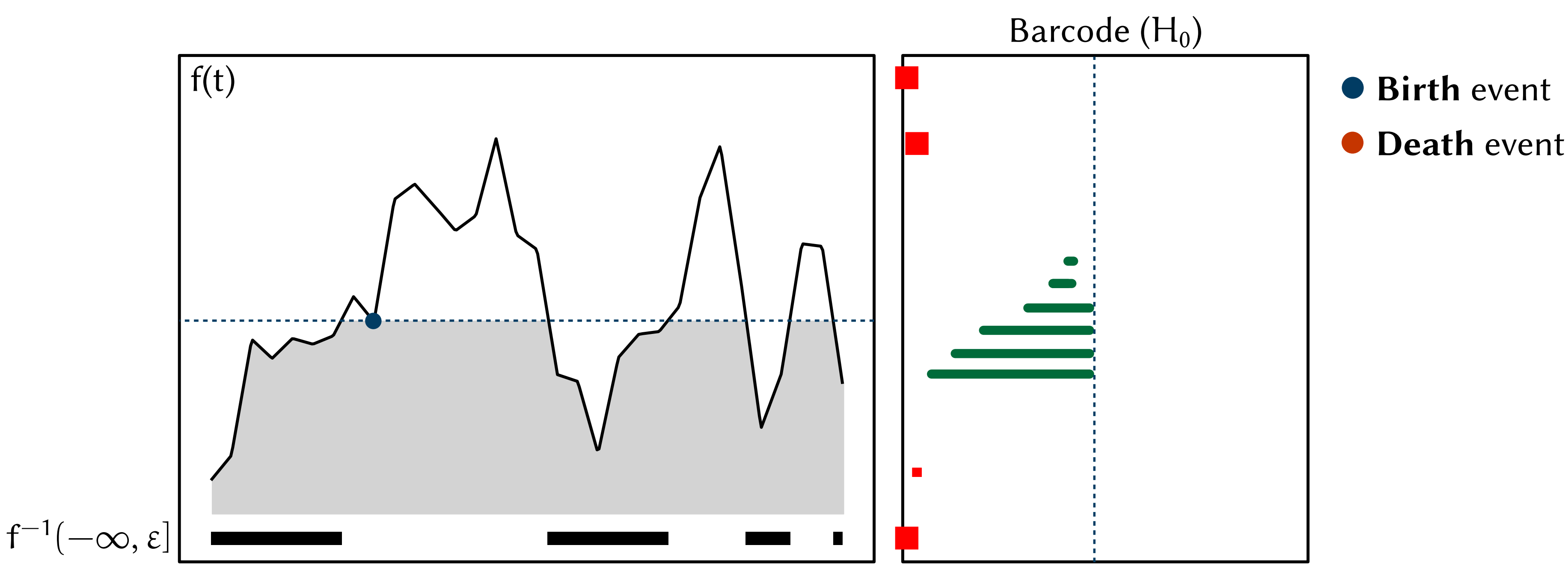

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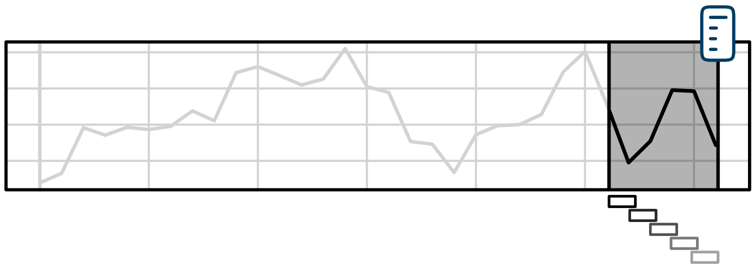

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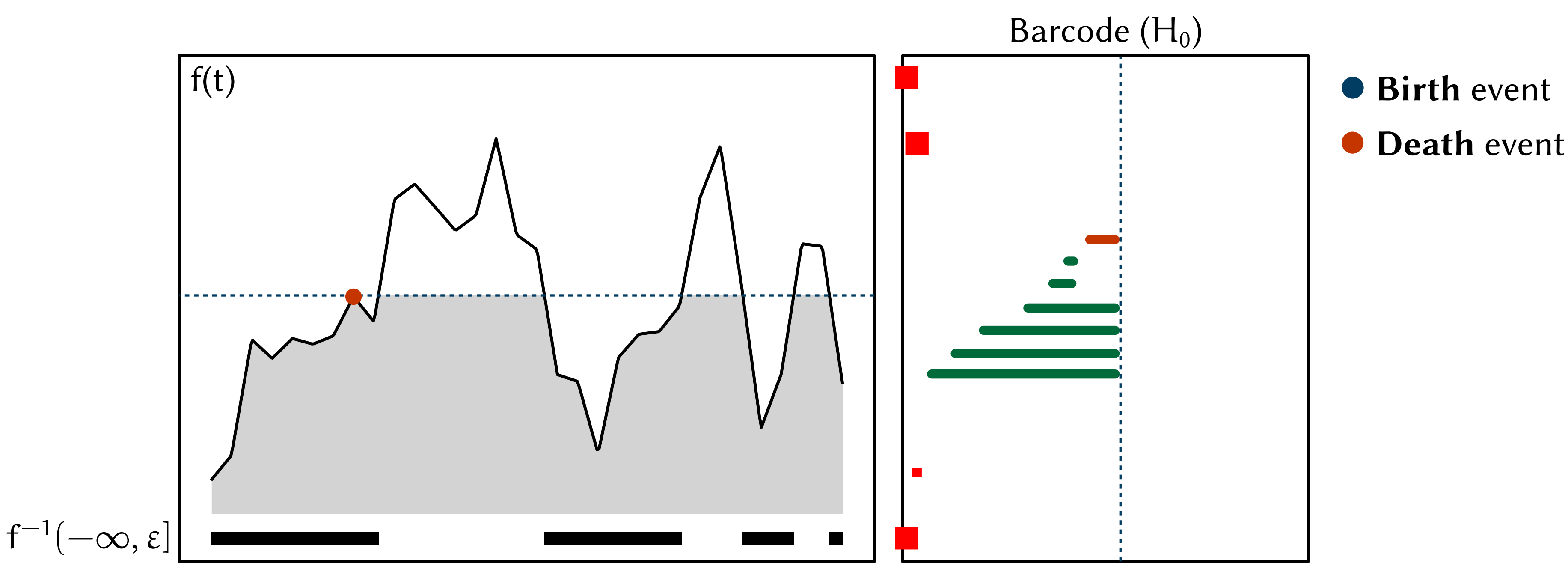

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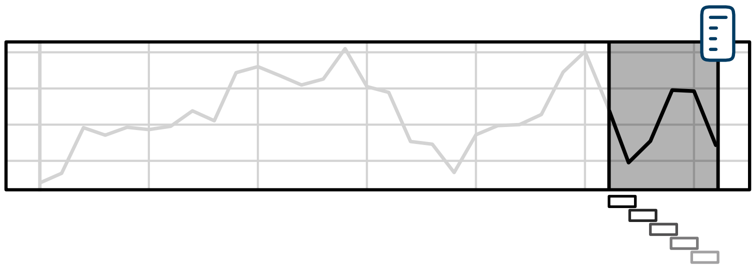

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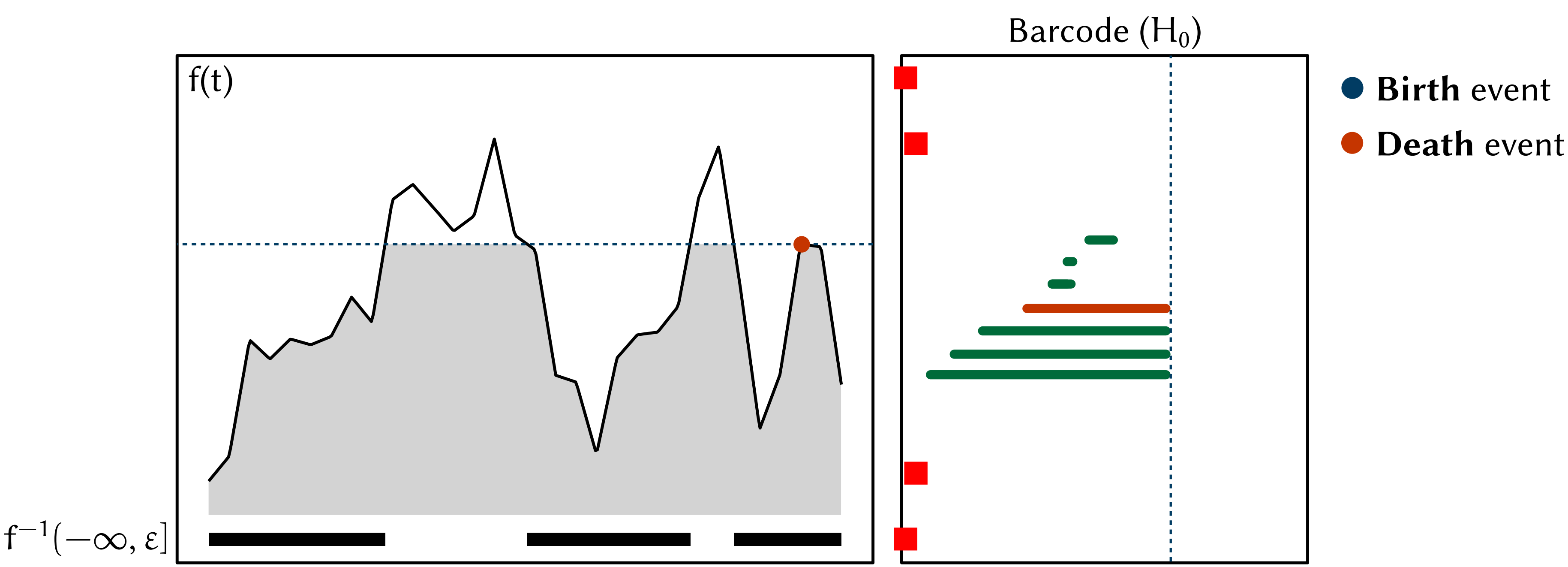

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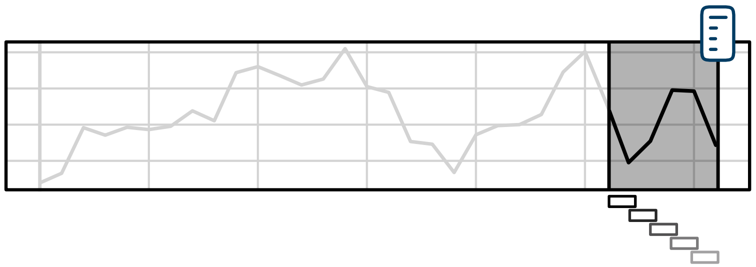

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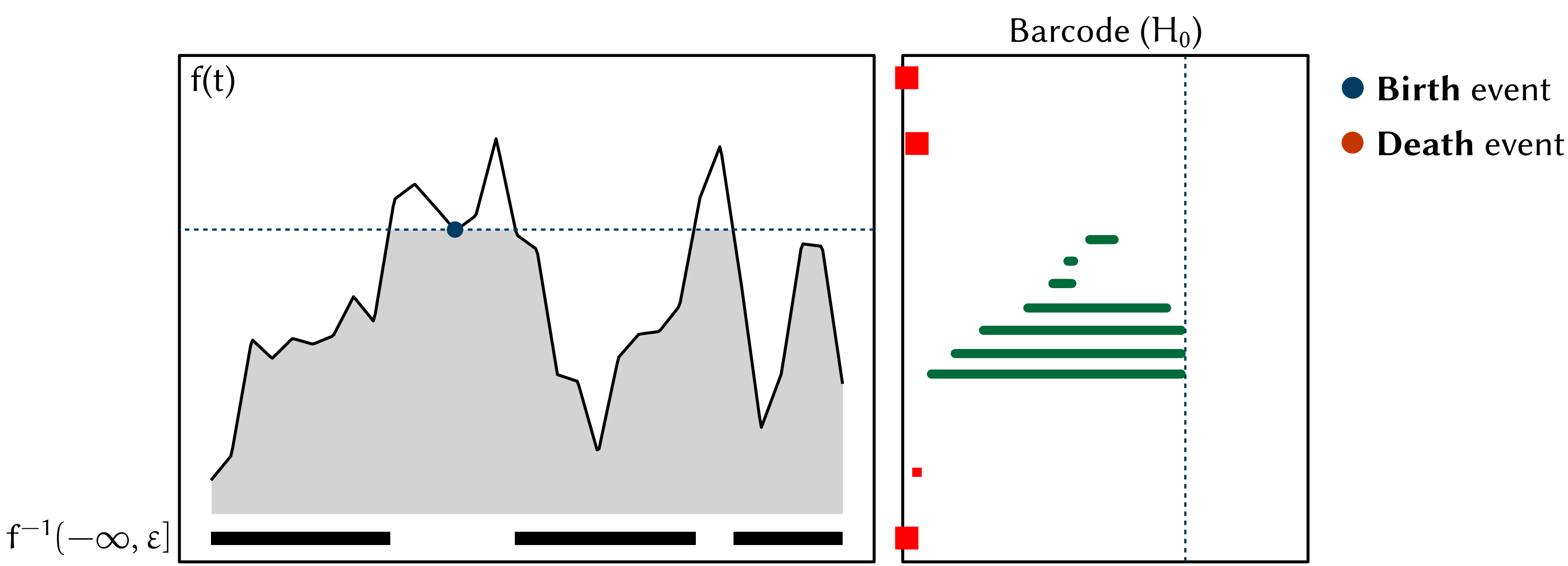

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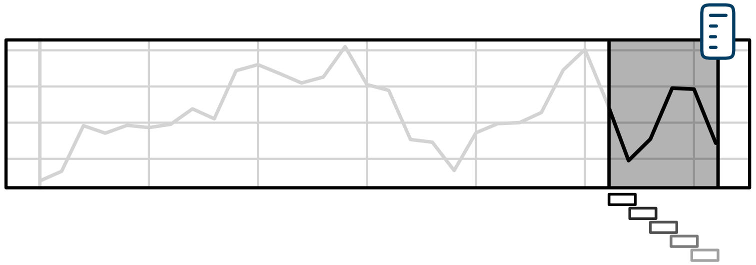

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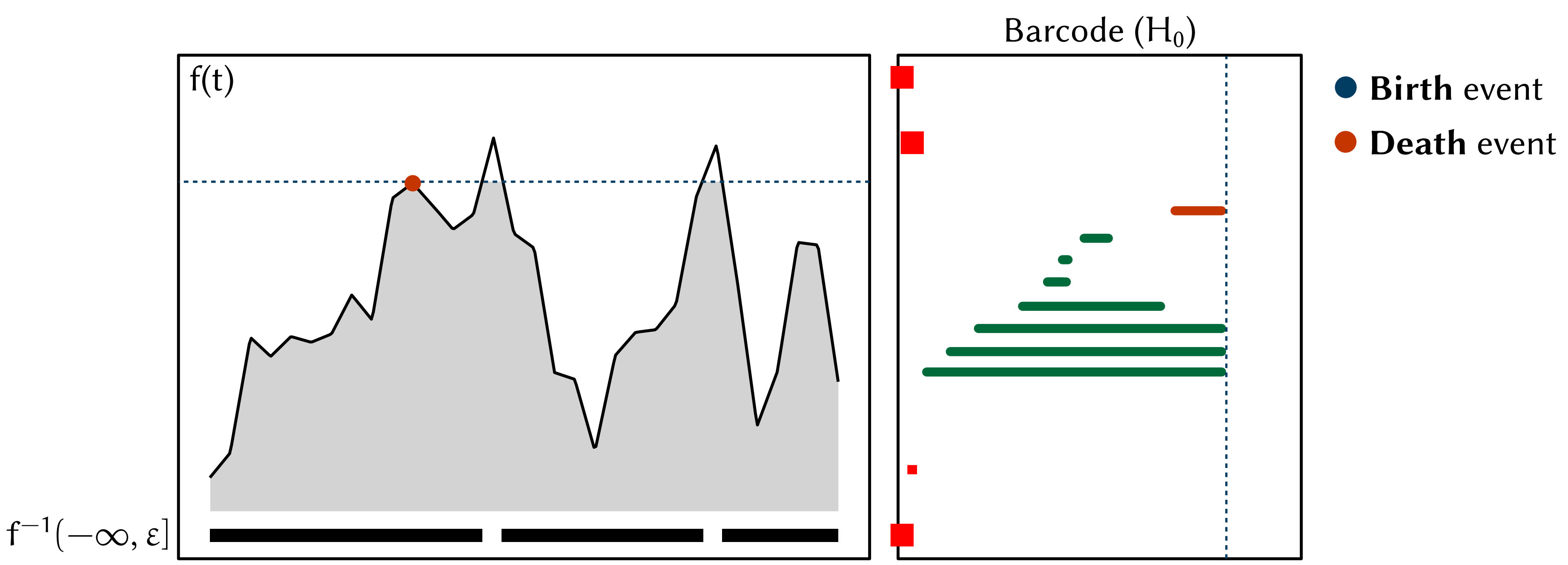

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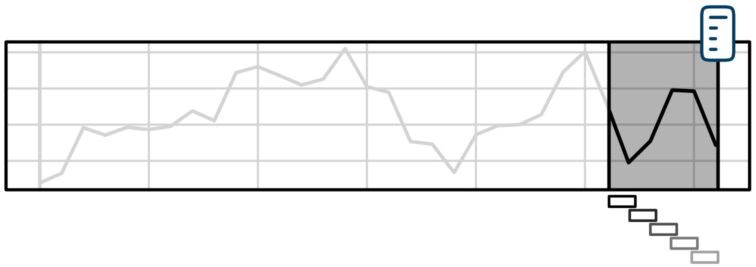

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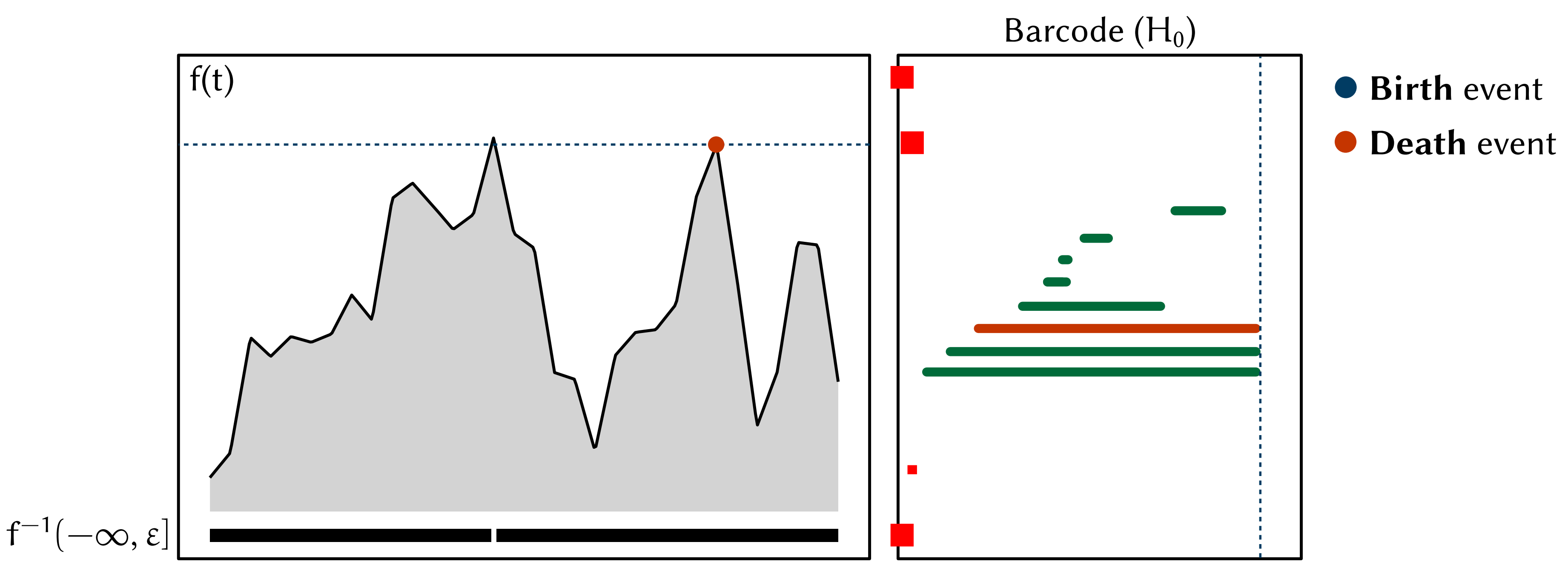

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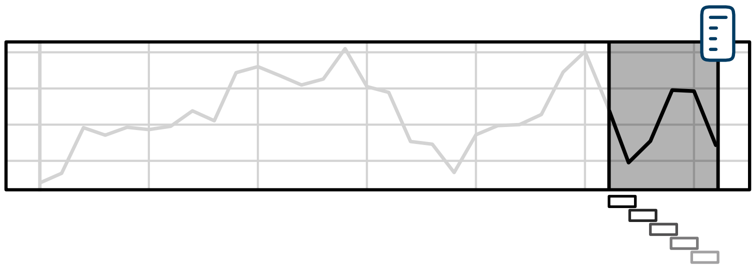

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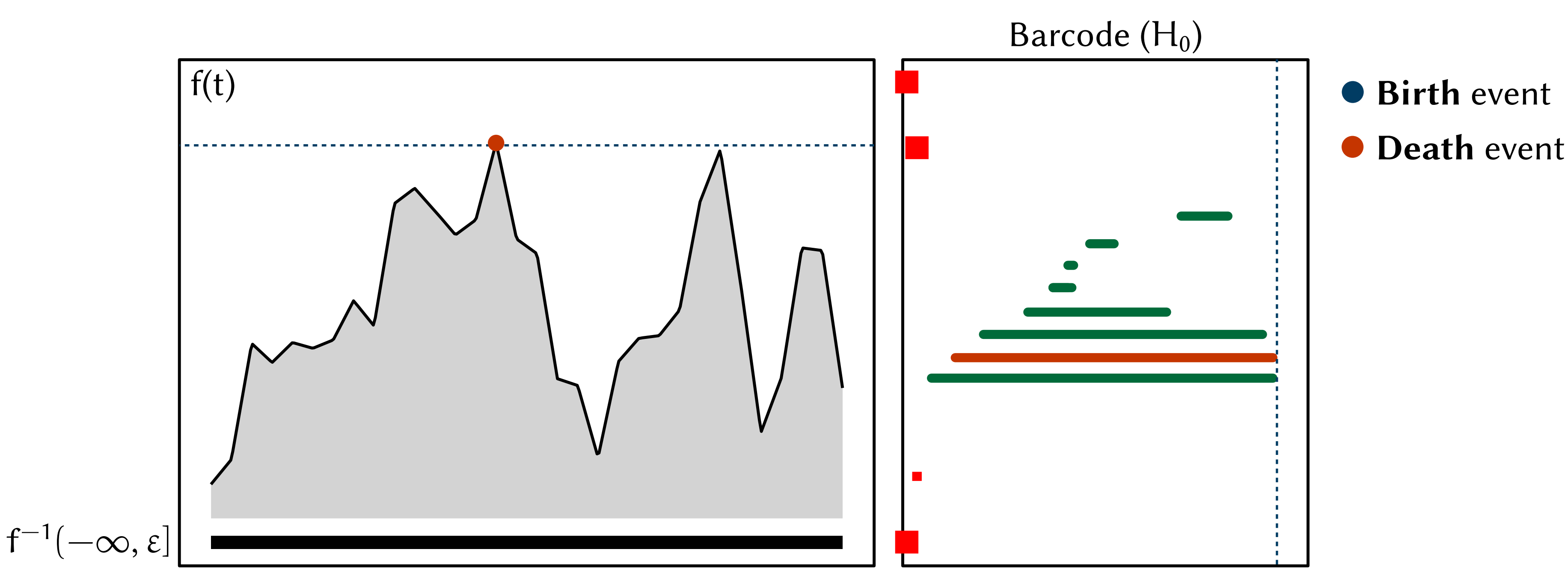

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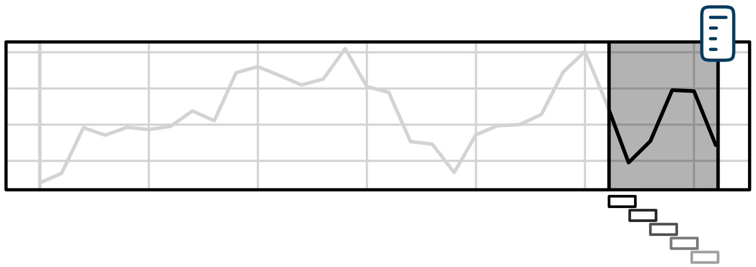

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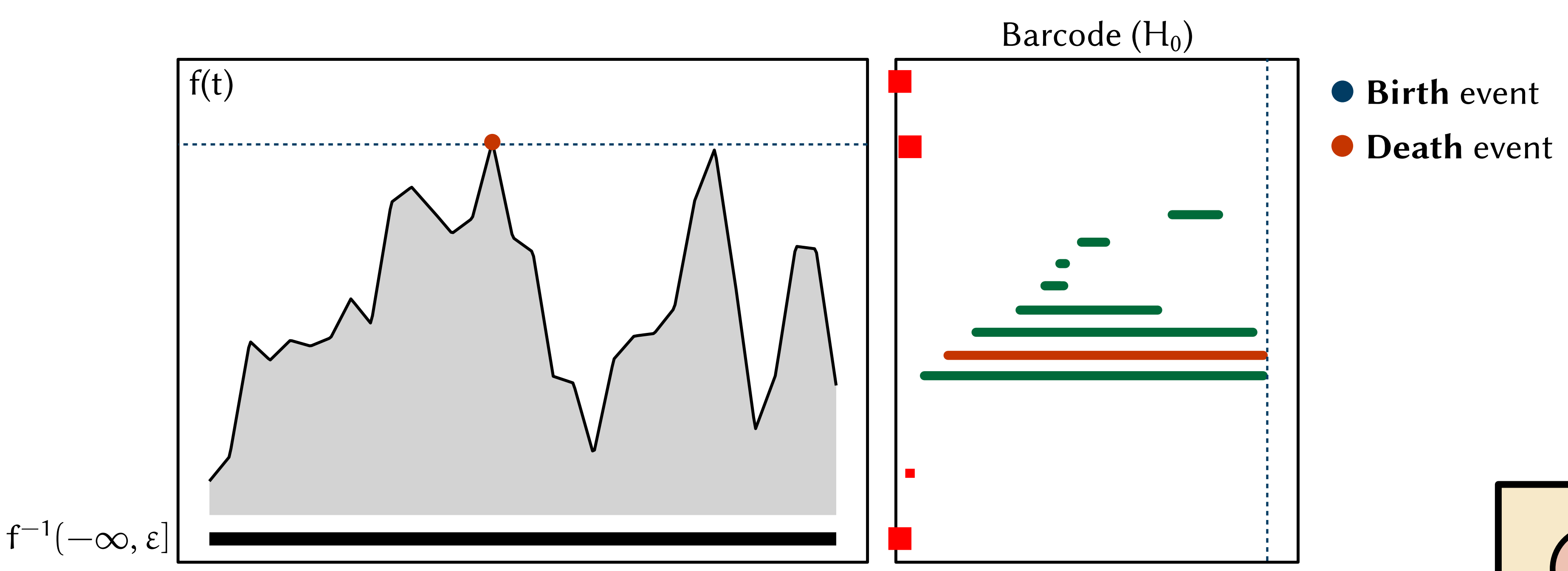

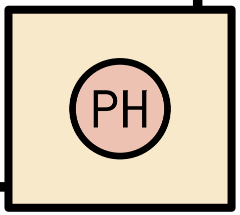

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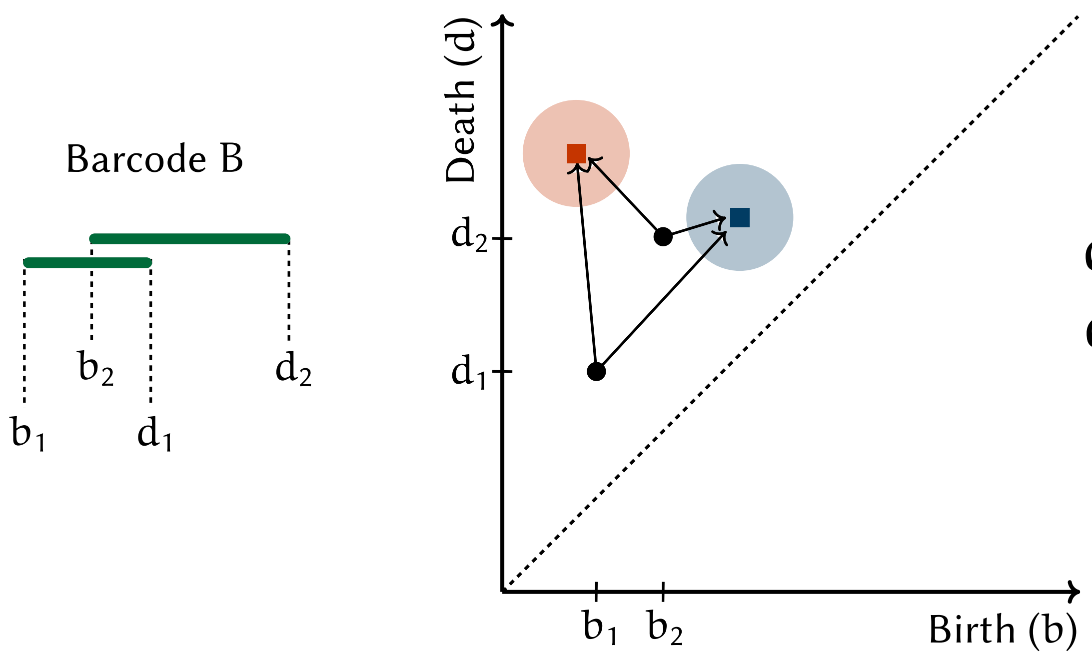
We will only consider 0-dimensional (**connectivity**) features in this work!

Vectorizing persistence barcodes

Barcodes are **multi-sets** of (birth, death) tuples.

To **vectorize** the multi-set we choose an approach from [Hofer et al., 2019]



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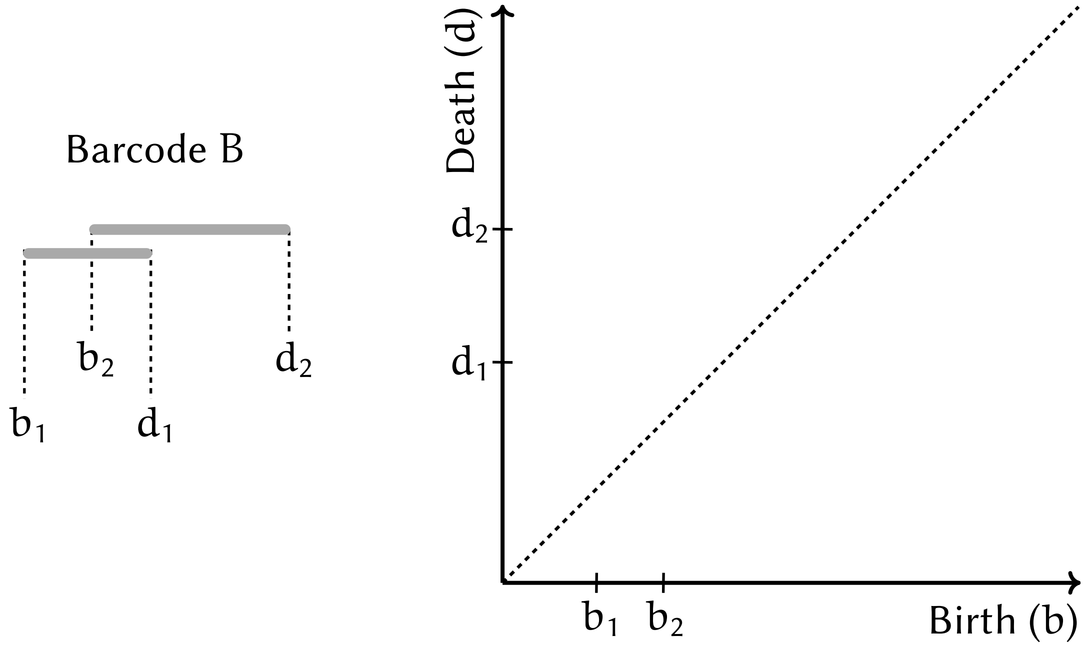

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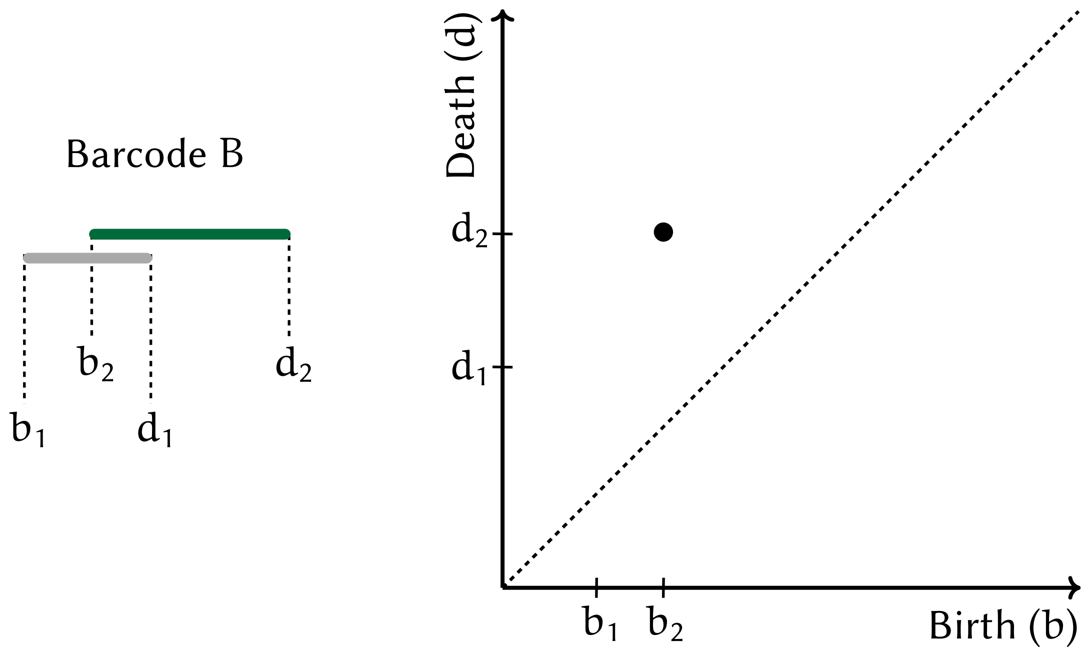

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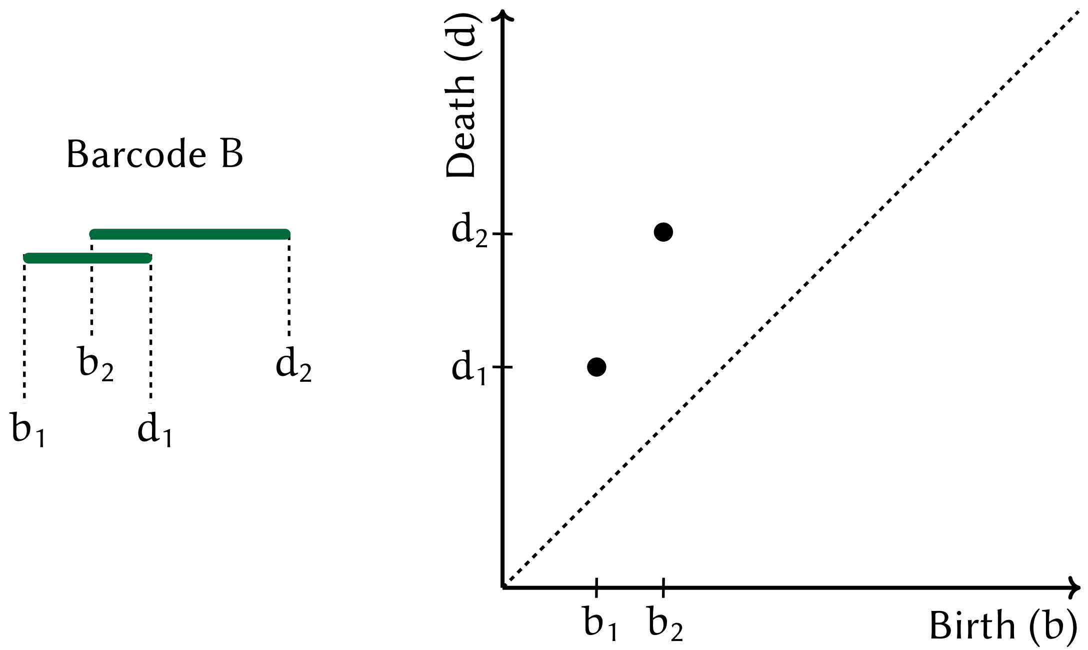

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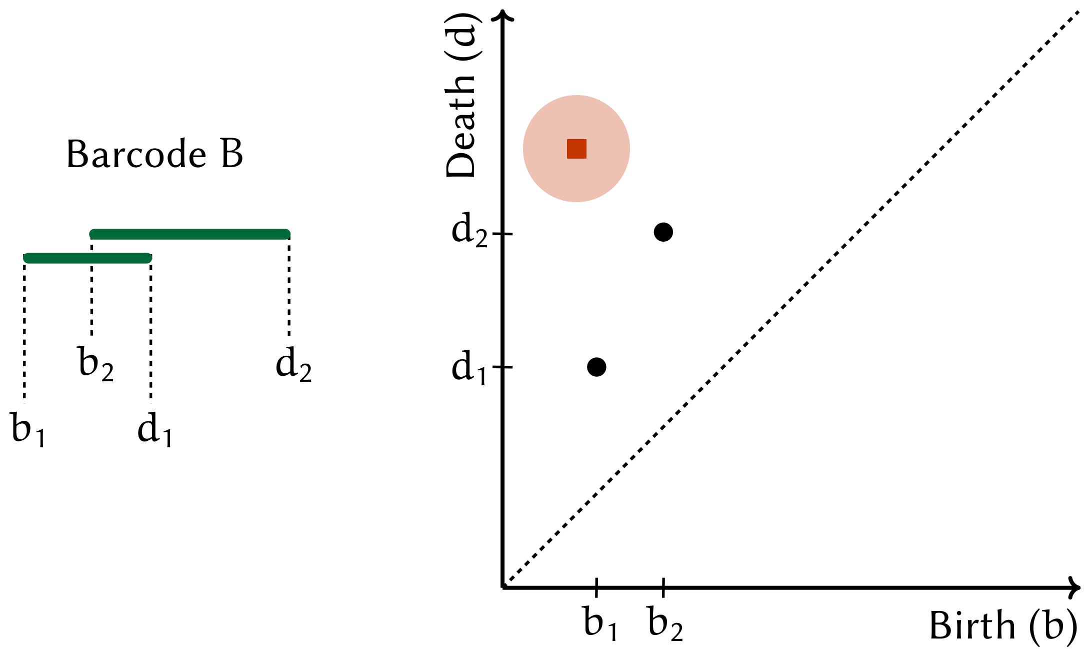

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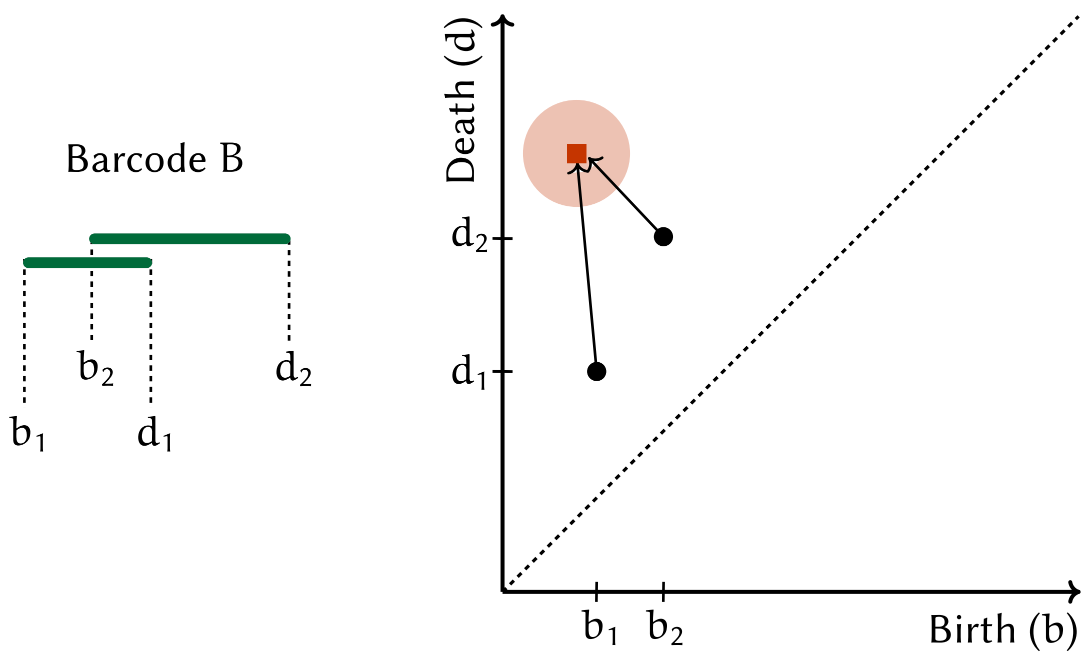

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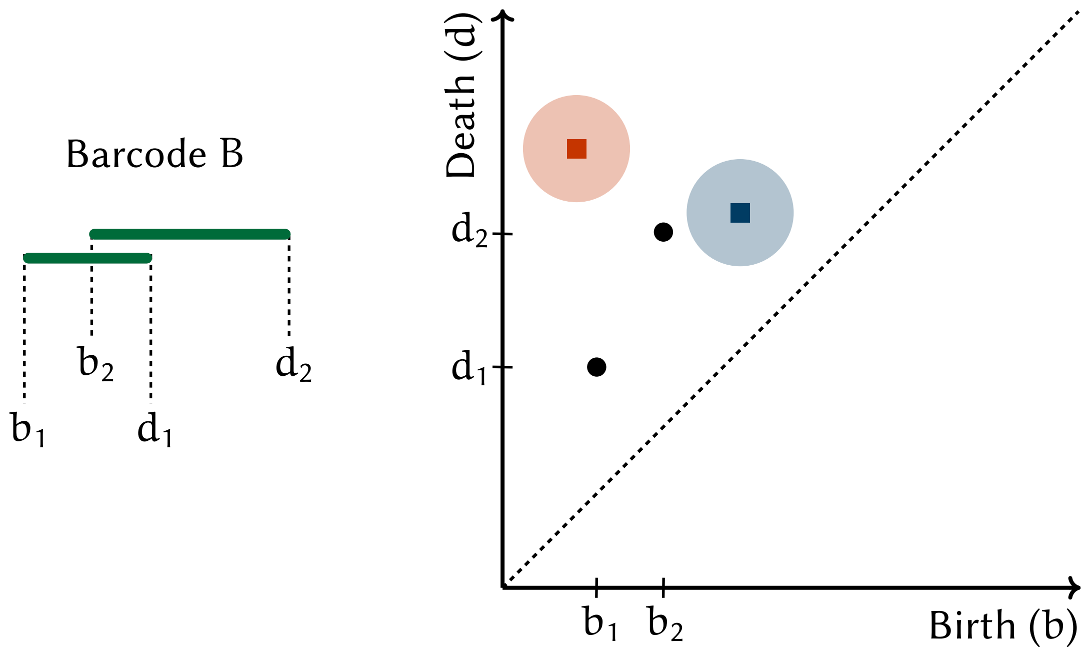

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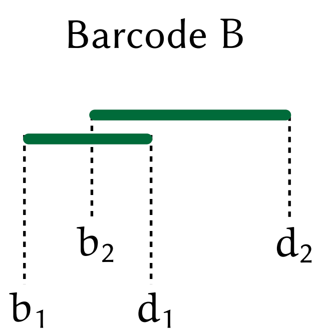

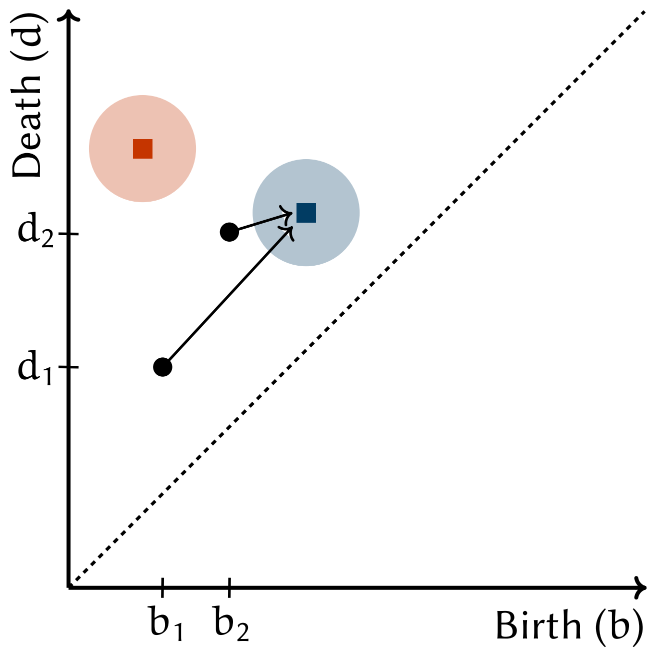

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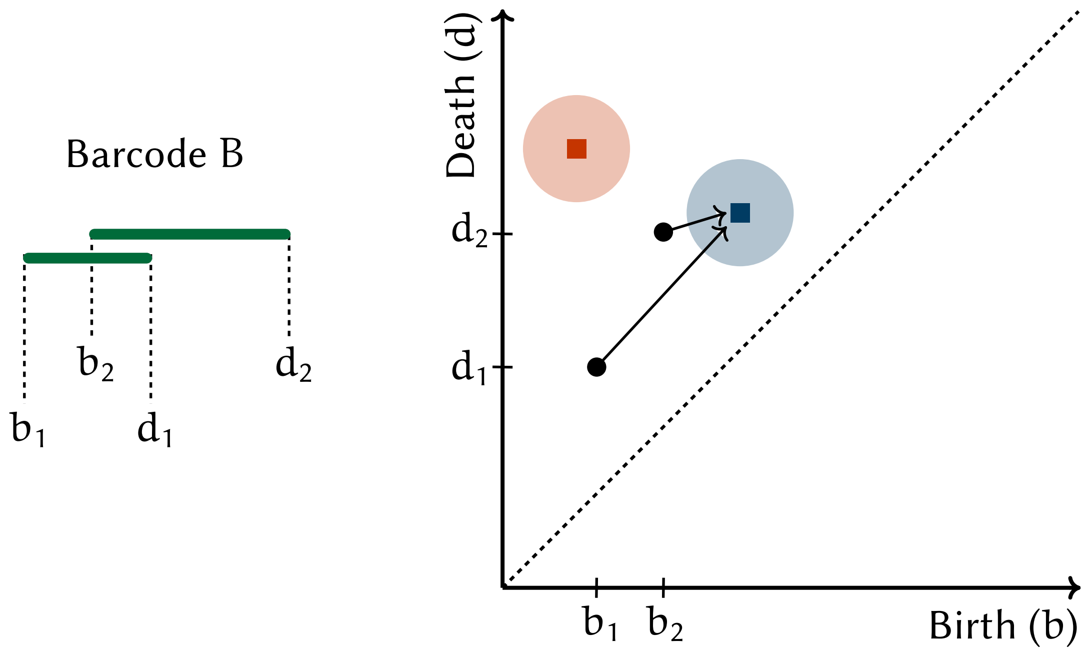

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Attending to local topological features

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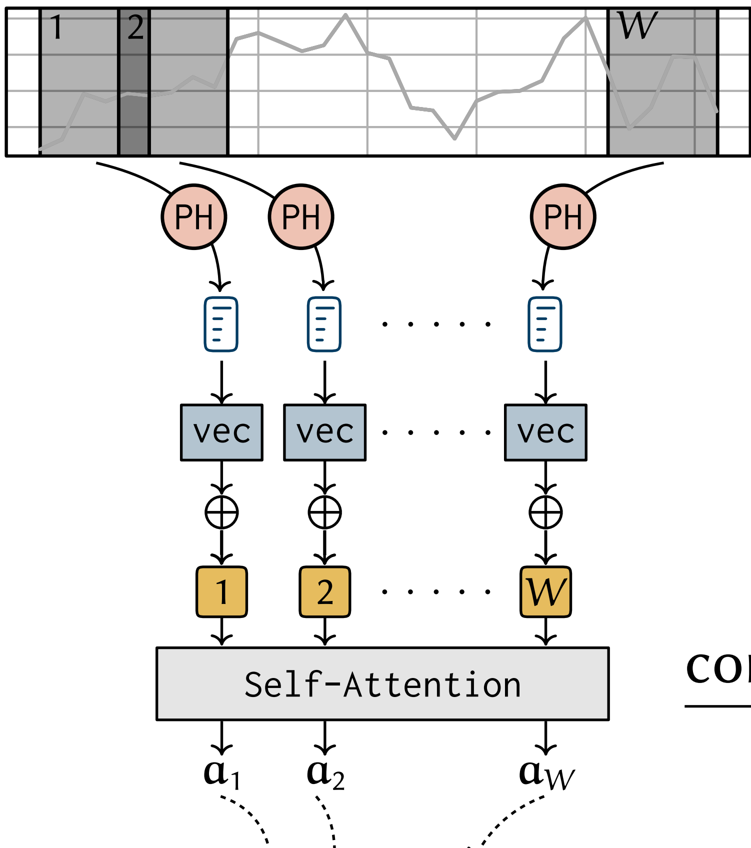

$$\boldsymbol{v} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_W)^\top \in \mathbb{R}^{We}$$

- PH: Persistent homology of time series function graph
- vec: Barcode vectorization (into $\mathbb{R}^e$)
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Topological Attention (TAN)

TAN

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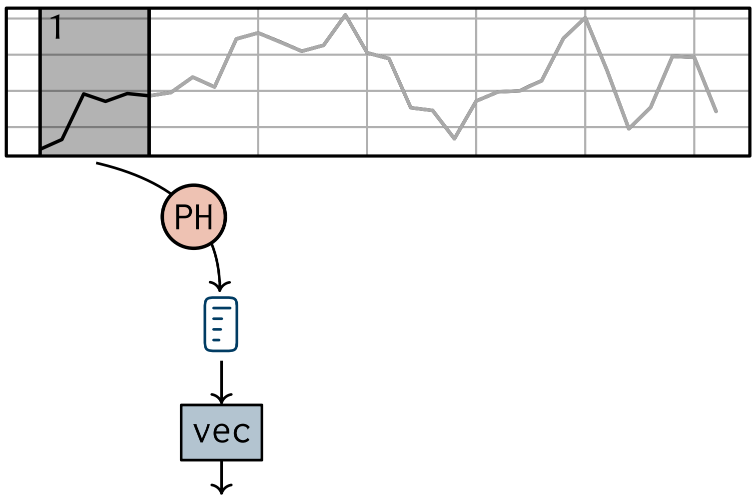

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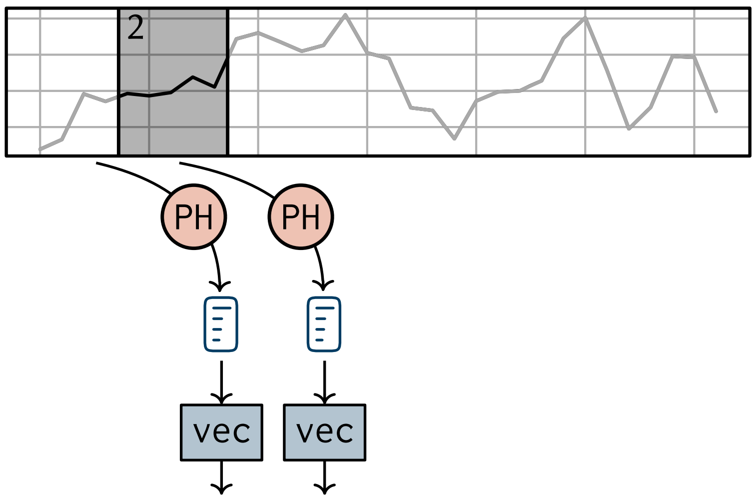

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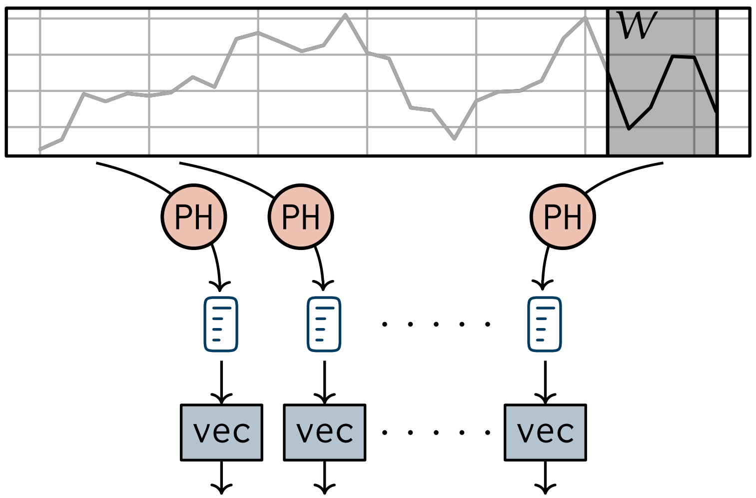

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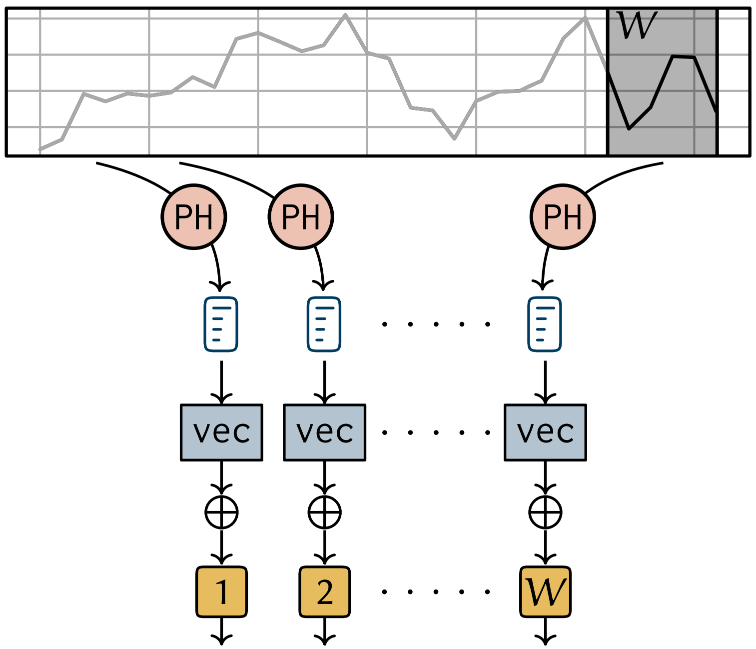

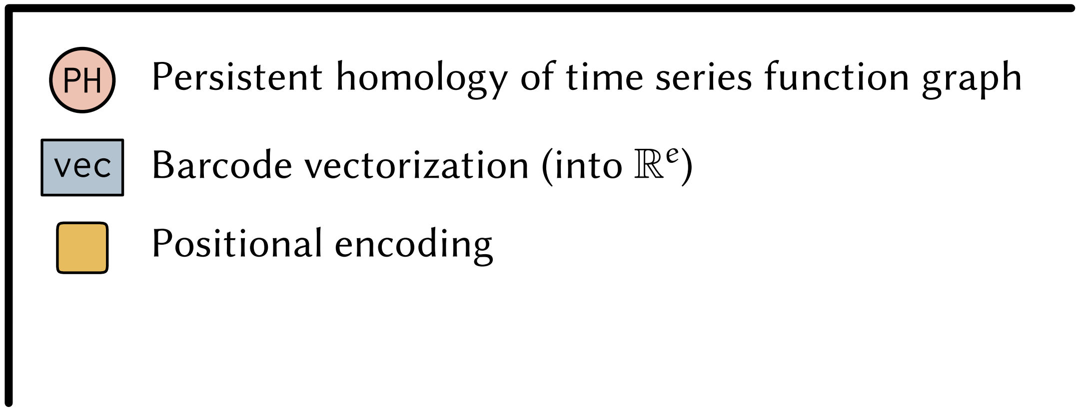

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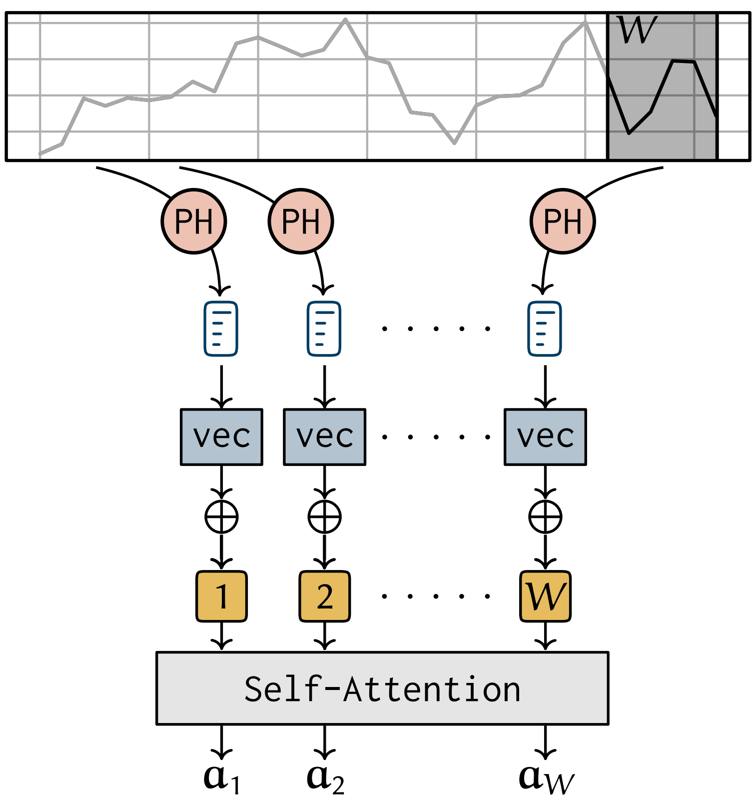

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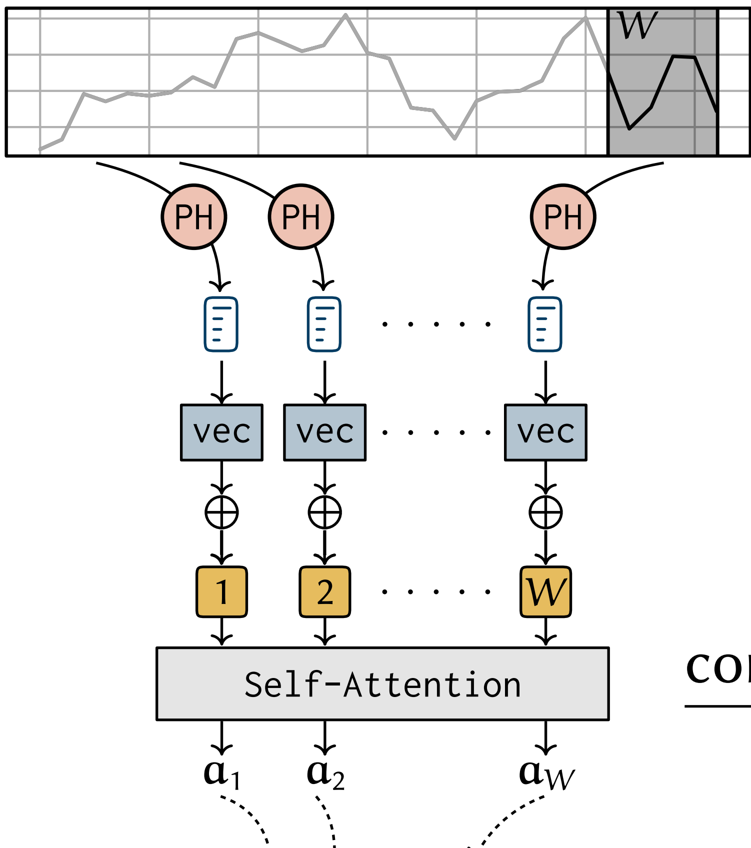

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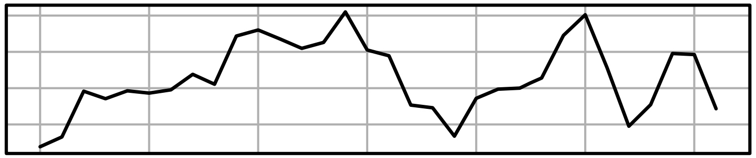

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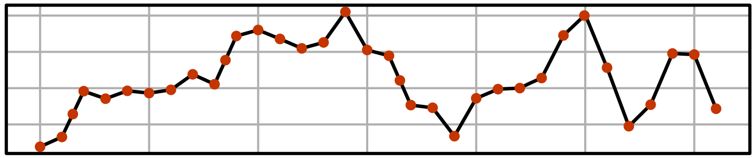

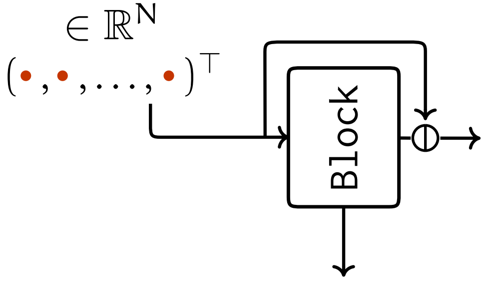

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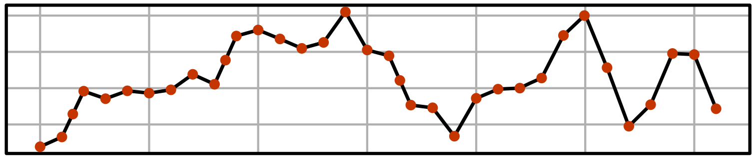

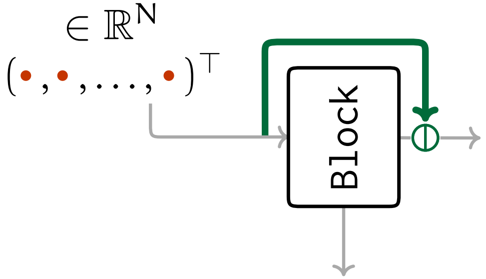

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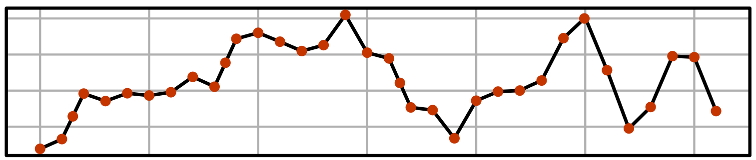

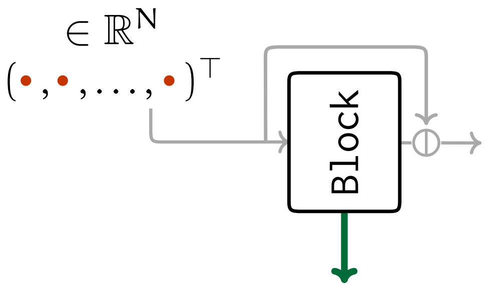

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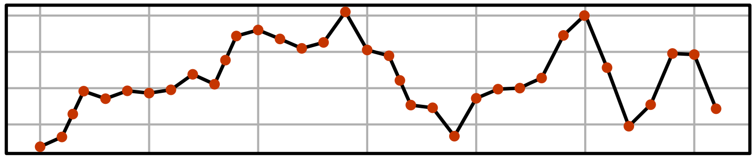

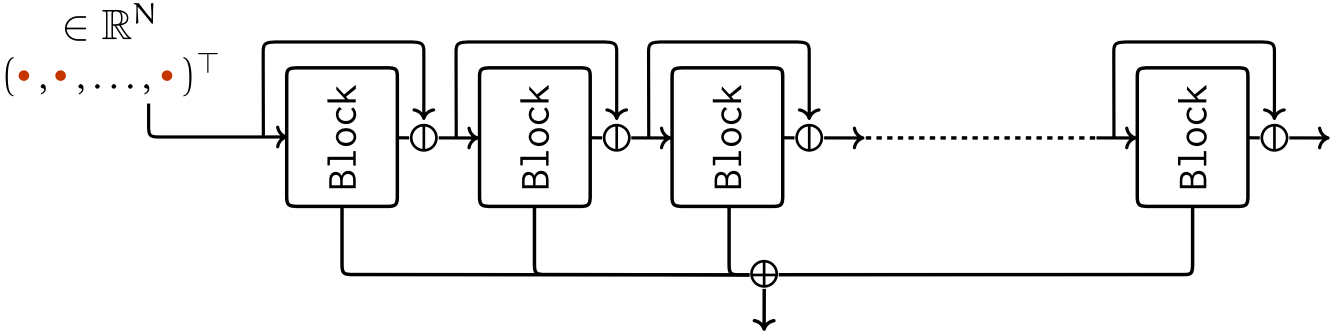

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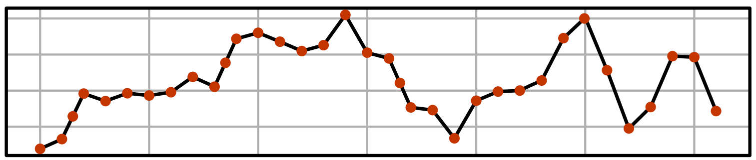

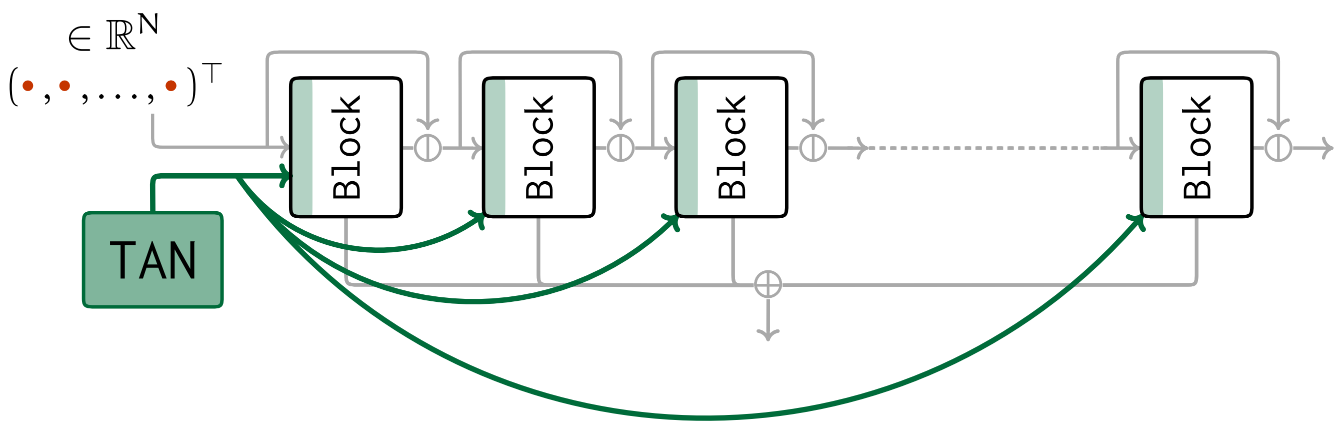

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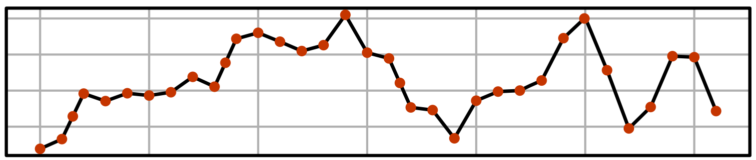

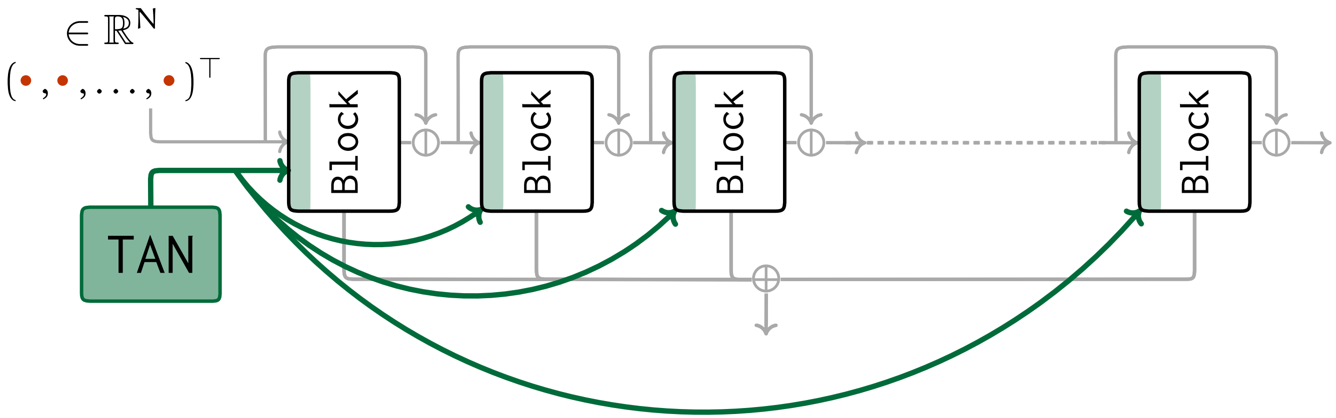

Setup:

- ▷ We follow the **ensemble** approach of [Oreshkin et al., 2019]
- ▷ Different (1) random seeds, (2) historical time horizons and (3) losses
- ▷ Forecast = **median** across ensemble forecasts

Experiments

Dataset: M4 competition data [Makridakis et al., 2018]

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- Fix evaluation protocol & scores (sMAPE, MASE, OWA)

Overall performance comparison (across all 100,000 time series):

Method	sMAPE $\downarrow$	OWA $\downarrow$
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†Benchmark	12.555	0.898
†Naive2	13.564	1.000

see paper for final results

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Next, we successively deactivate **TAN** parts (on a smaller ensemble):

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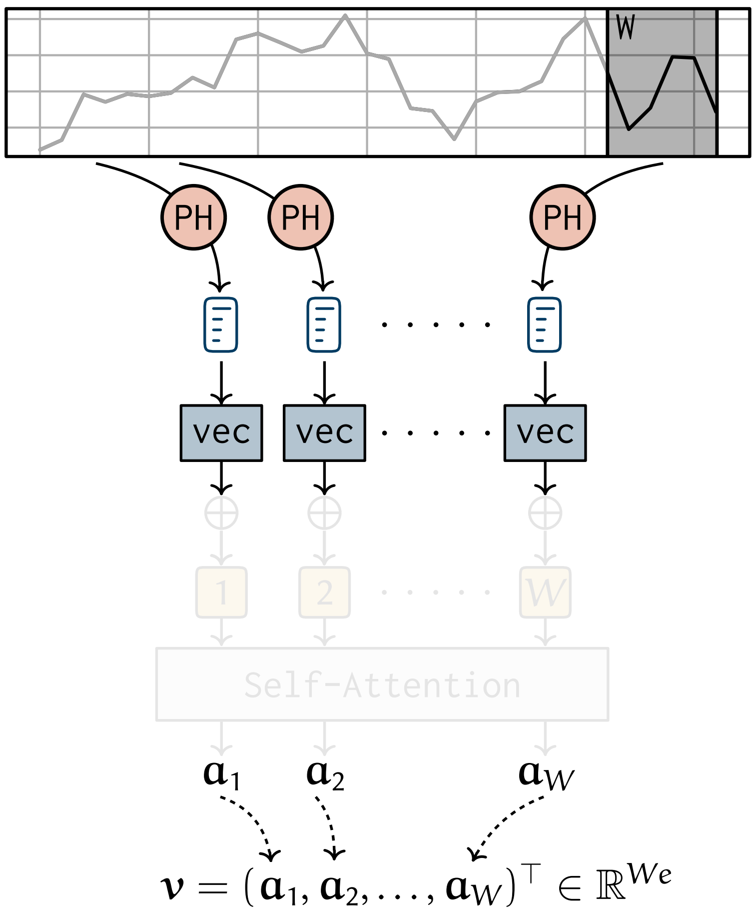

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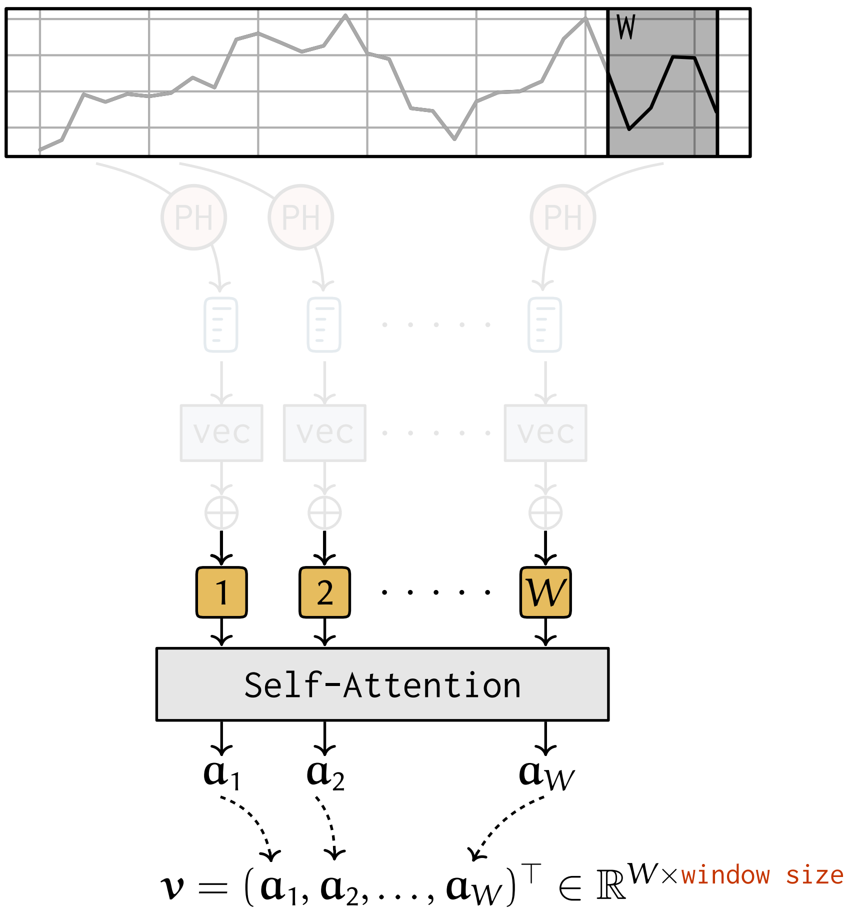

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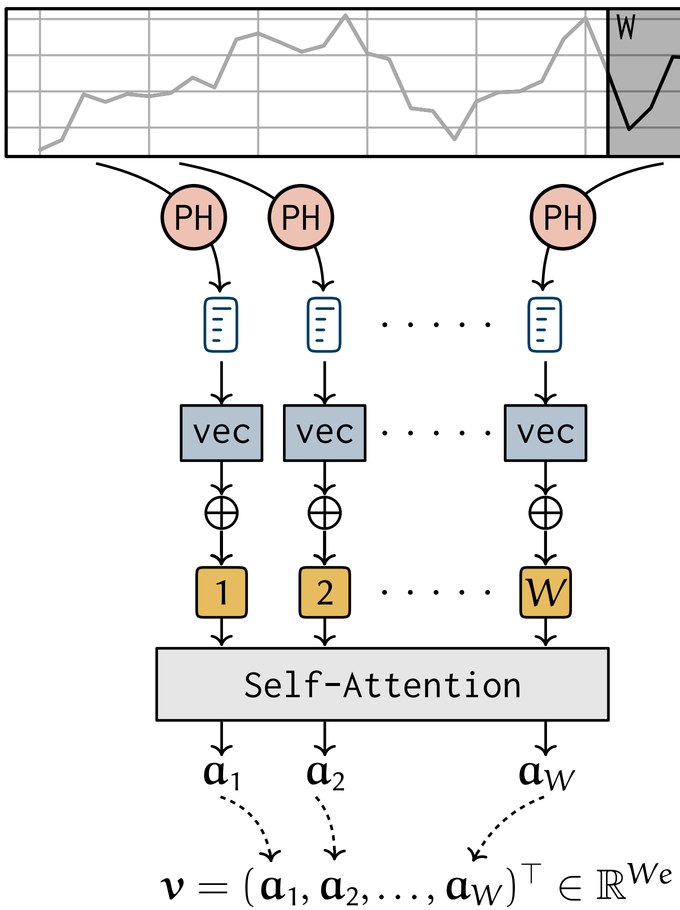

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Experiments – Runtime

Runtime of 0-dim. persistent homology as a function of window size m:

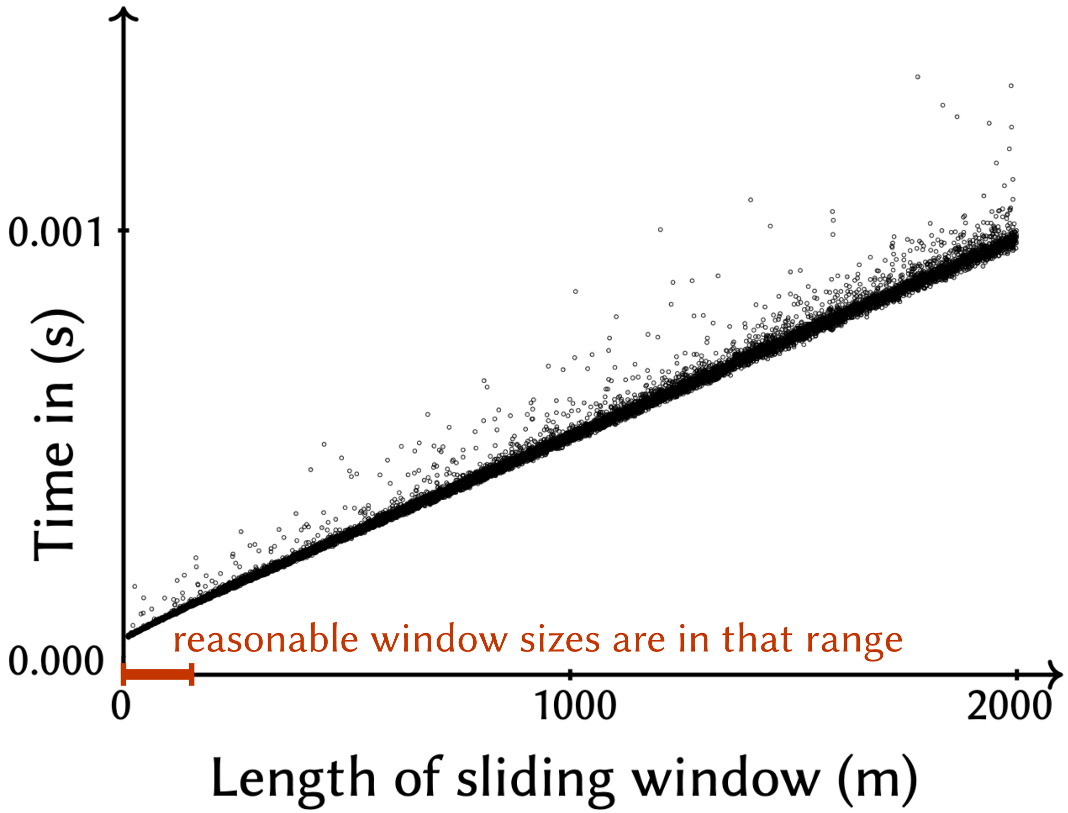
Complexity: $\mathcal{O}(m\alpha^{-1}(m))$

α^{-1} = inverse Ackerman function

m = window size

Experiments – Runtime

Runtime of 0-dim. persistent homology as a function of window size m:



Complexity: $\mathcal{O}(m\alpha^{-1}(m))$

α^{-1} = inverse Ackerman function

m = window size

Conclusion

We propose an approach ...

- ▷ . . . to attend to local topological time series features
- ▷ . . . that is easy to integrate at moderate computational overhead
- ▷ . . . that supplies complementary information beyond raw observations

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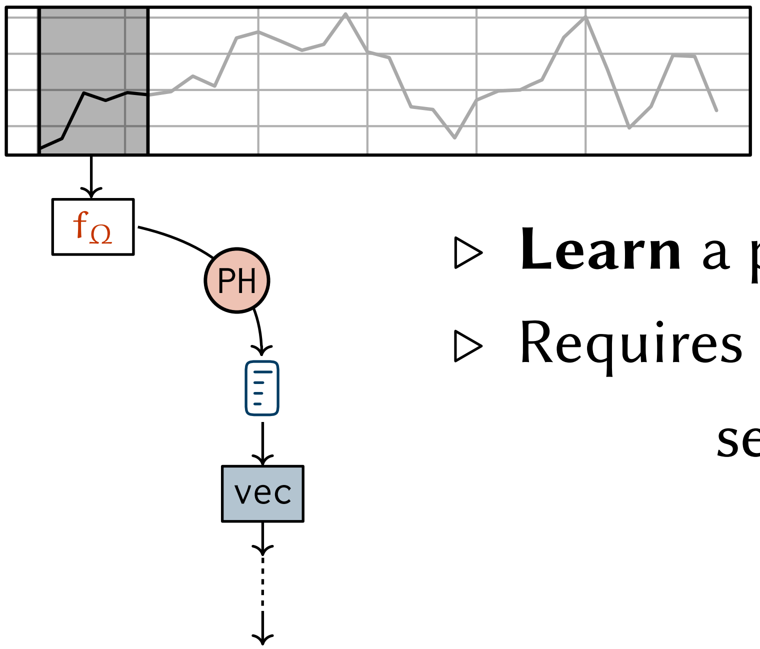
Possible future direction



- ▷ **Learn** a parameterized "filter" f_Ω for PH
- ▷ Requires to backprop through PH computation

see [Hofer et al., 2020], [Carrière et al.,2021]

Thank You!

Source code & Slides are available at

https://github.com/plus-rkwitt/TAN