UniXGrad: A Universal, Adaptive Algorithm with Optimal Guarantees for Constrained Optimization

Ali Kavis[1]
EPFL

Kfir Y. Levy[1]
Technion

Francis Bach
INRIA

Volkan Cevher
EPFL

NeurIPS 2019

December 10, 2019

[1]Equal contribution

Introduction

Problem Definition

$$\min_{x \in \mathcal{K}} f(x) \tag{1}$$

$f : \mathcal{K} \to \mathbb{R}$ is a convex function

$\mathcal{K} \subset \mathbb{R}^d$ is a **compact**, convex set

Motivation

Aim of this work

- **Universal:** Optimal for smooth and non-smooth problems
- **Adaptive:** No knowledge of Lipschitz constant and variance
- **Constrained:** Extend existing results to constrained problems

Our algorithm: UniXGrad

Algorithm 1 UniXGrad

Input: # of iterations T , $y_0 \in K$, weight $\alpha_t = t$, learning rate $\{\eta_t\}_{t\in[T]}$

1: **for** $t = 1, ..., T$ **do**

2: $\quad x_t = \arg\min_{x\in\mathcal{K}} \alpha_t \langle x, M_t\rangle + \frac{1}{\eta_t} D_{\mathcal{R}}(x, y_{t-1}), \qquad M_t = \nabla f(\tilde{z}_t)$

3: $\quad y_t = \arg\min_{y\in\mathcal{K}} \alpha_t \langle y, g_t\rangle + \frac{1}{\eta_t} D_{\mathcal{R}}(y, y_{t-1}), \qquad g_t = \nabla f(\bar{x}_t)$

4: **end for**

$$\bar{x}_t = \frac{\alpha_t x_t + \sum_{i=1}^{t-1} \alpha_i x_i}{\sum_{i=1}^{t} \alpha_i} \qquad \tilde{z}_t = \frac{\alpha_t y_{t-1} + \sum_{i=1}^{t-1} \alpha_i x_i}{\sum_{i=1}^{t} \alpha_i} \tag{2}$$

$$\eta_t = \frac{2D}{\sqrt{1 + \sum_{i=1}^{t-1} \alpha_i^2 \|g_i - M_i\|_*^2}} \tag{3}$$

Conversion Scheme and Adaptive Bounds

$$\textbf{Weighted Regret: } R_T(x_*) = \sum_{t=1}^{T} \alpha_t \left\langle \nabla f(\bar{x}_t), x_t - x_* \right\rangle$$

Adaptive bound

$$R_T(x_*) \leq \frac{7}{2} D \sqrt{1 + \sum_{t=1}^{T} \alpha_t^2 \left\| g_t - M_t \right\|_\star^2} - \frac{D}{2} \tag{4}$$

Lemma (Regret $\Rightarrow$ Rate)

$$f(\bar{x}_T) - f(x_*) \leq \frac{2R_T(x_*)}{T^2}. \tag{5}$$

Convergence in Non-smooth Setting

Theorem

If f is G-Lipschitz, Algorithm 1 guarantees

$$E\left[f(\bar{x}_T)\right] - \min_{x \in \mathcal{K}} f(x) \leq \frac{6D}{T^2} + \frac{14GD}{\sqrt{T}}. \tag{6}$$

Remark

- Regret analysis is agnostic to definitions of g_t and M_t.
- Conversion scheme requires $g_t = \nabla f(\bar{x}_t)$.

Convergence in Smooth Setting

Theorem

If f is L-smooth and oracle is **deterministic**, *Algorithm 1 ensures*

$$f(\bar{x}_T) - \min_{x \in \mathcal{K}} f(x) \leq \frac{20\sqrt{7}D^2L}{T^2}. \tag{7}$$

If oracle is **stochastic**,

$$\mathbb{E}\left[f(\bar{x}_T)\right] - \min_{x \in \mathcal{K}} f(x) \leq \frac{224\sqrt{14}D^2L}{T^2} + \frac{14\sqrt{2}\sigma D}{\sqrt{T}}. \tag{8}$$

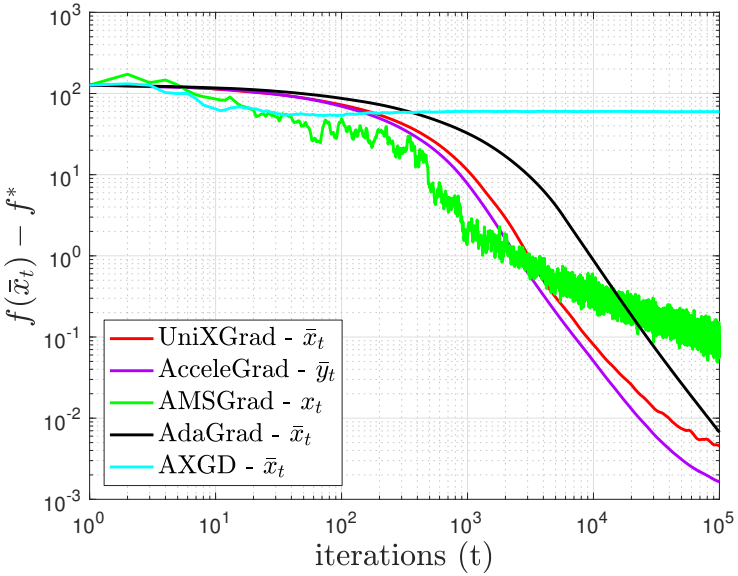
Remark

f is L-smooth $\Rightarrow$ bounded gradients are not required.

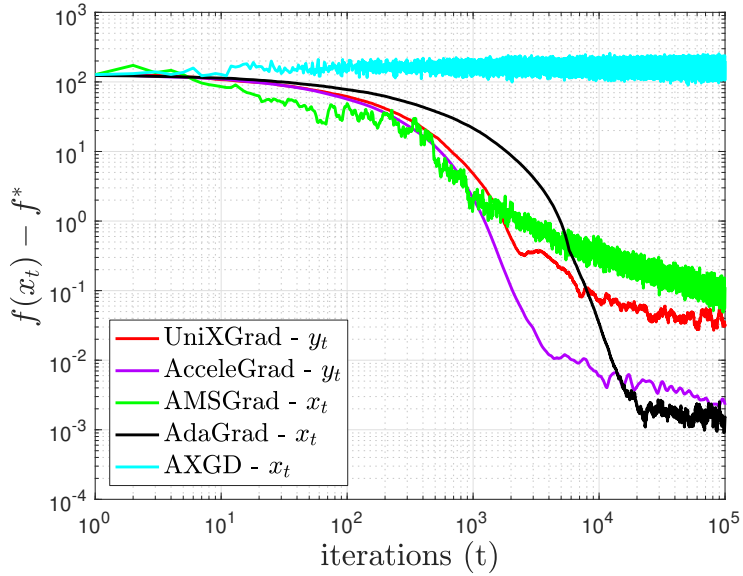
Convergence Behavior

Least-squares with ℓ_2 norm ball constraint:

$$\min_{\|x\|_2 < r} \frac{1}{2n} \|Ax - b\|_2^2 ,$$



(a) Average Iterate

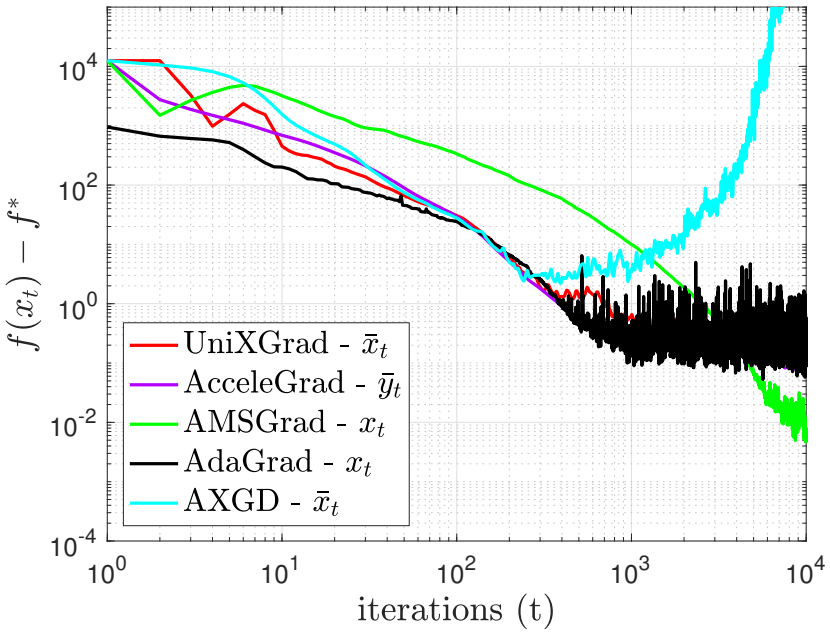


(b) Last Iterate

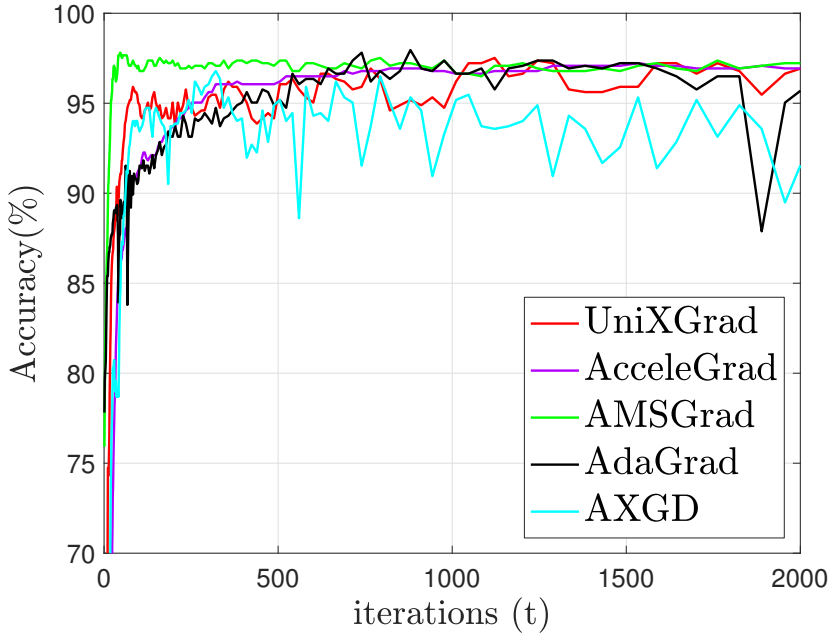
Figure 1: Convergence in the **stochastic** oracle setting, $x_* \in \text{Boundary}(\mathcal{K})$

SVM Classification

- SVM with squared Hinge loss and ℓ_2 regularization
- breast-cancer data from libsvm dataset, 80/20 train/test ratio
- Training batch size: 5, number of runs: 5.



(a) Convergence w.r.t. training data



(b) Test Accuracy

Figure 2: SVM classification using breast-cancer data (Chang and Lin, 2011)

References I

Chang, Chih-Chung and Chih-Jen Lin (2011). "LIBSVM: A library for support vector machines". In: *ACM Transactions on Intelligent Systems and Technology* 2 (3), 27:1–27:27.