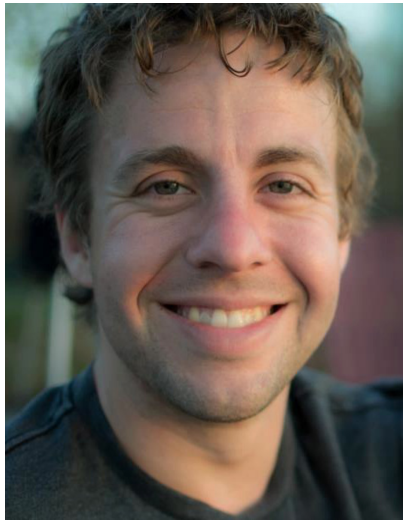
Principal Component Projection and Regression in Nearly Linear Time through Asymmetric SVRG

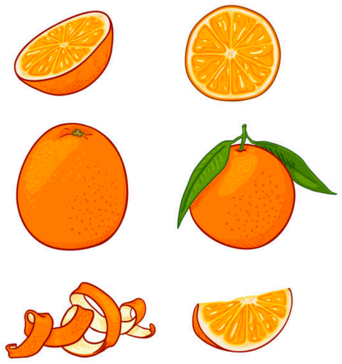
Yujia Jin, Aaron Sidford



Stanford University

How to make juice?

Peel & cut



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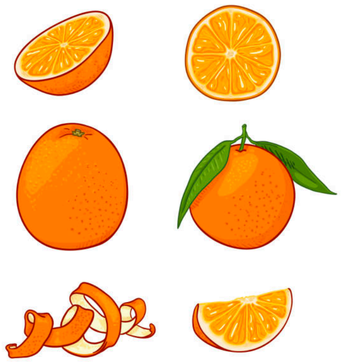

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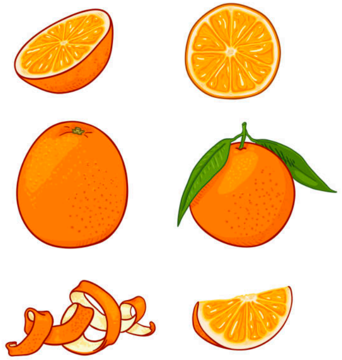

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Juice pulper

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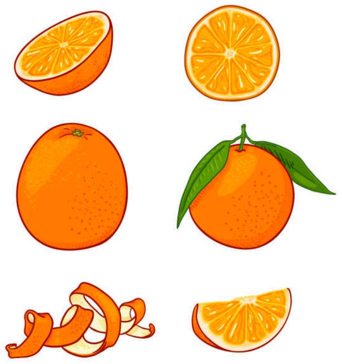

Put in juicer

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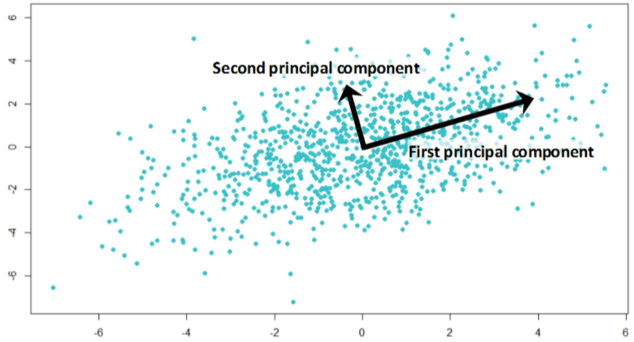
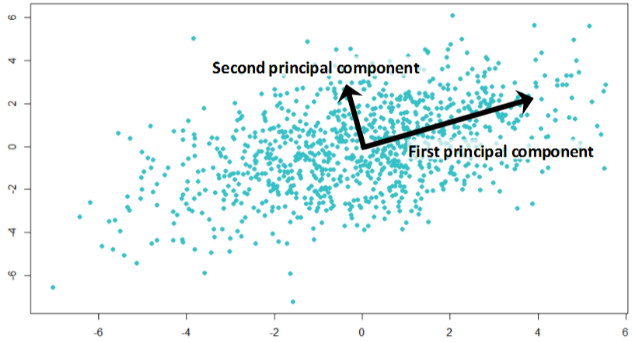
Efficient!

Motivation



Data preprocessing: PCA

Motivation



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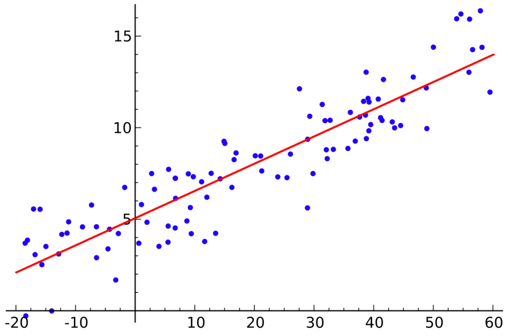

Data analysis tasks: Projection, regression, etc.

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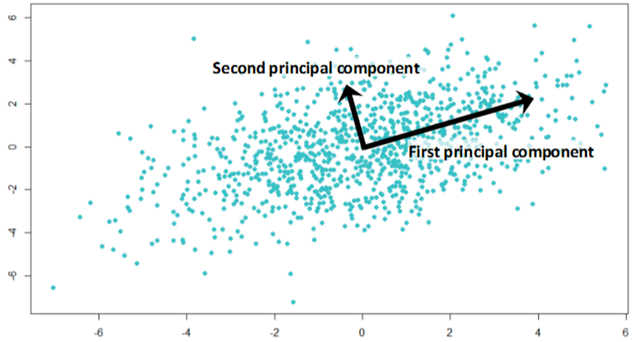

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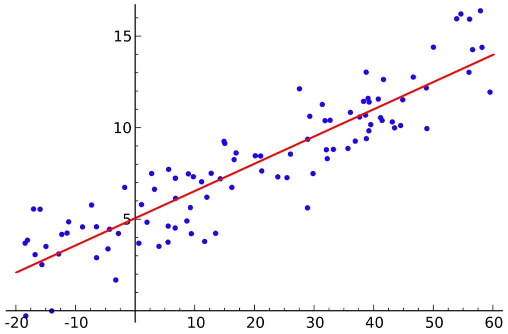

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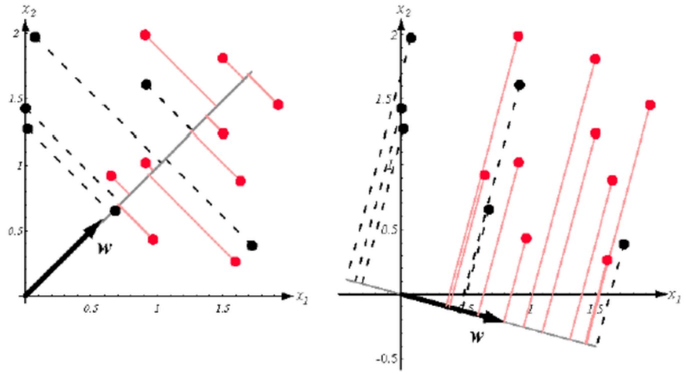
Principal Component Projection & Regression



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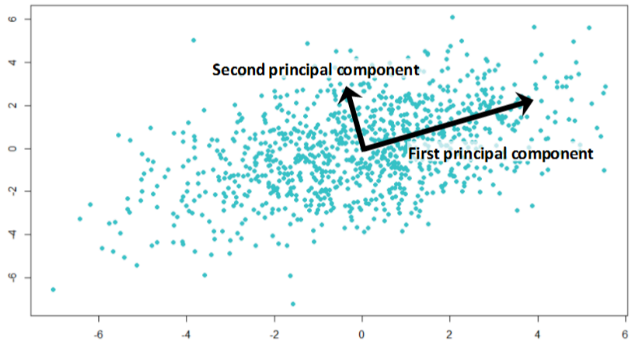

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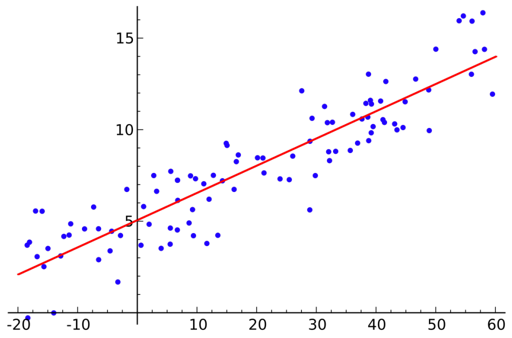

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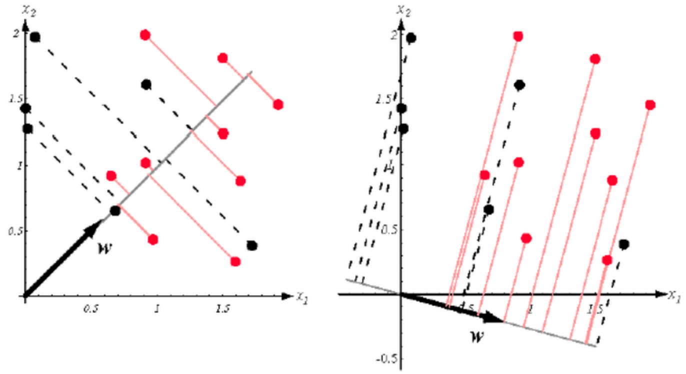
Principal Component
Projection & Regression



Combining tasks
in one-step

Problems: PCP and PCR

A - n-by-d data matrix;
$\mathcal{V}^\star$ - interested top-eigenspace;
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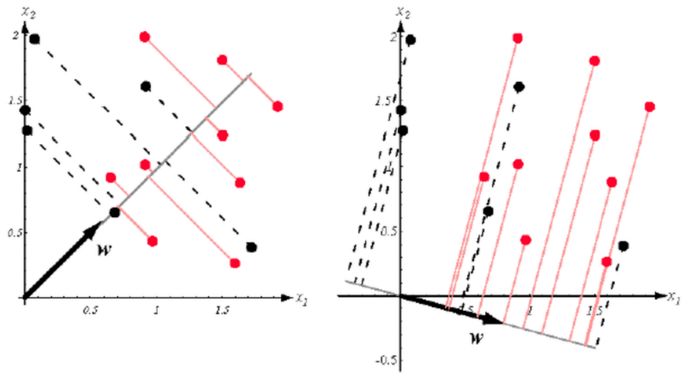

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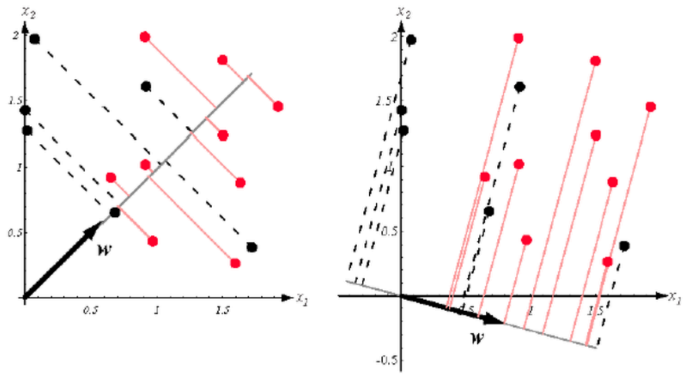

$$\text{PCR} \xrightarrow[\text{[FMMS '16]}]{\tilde{O}(1)\text{ poly. red.}} \text{PCP}$$

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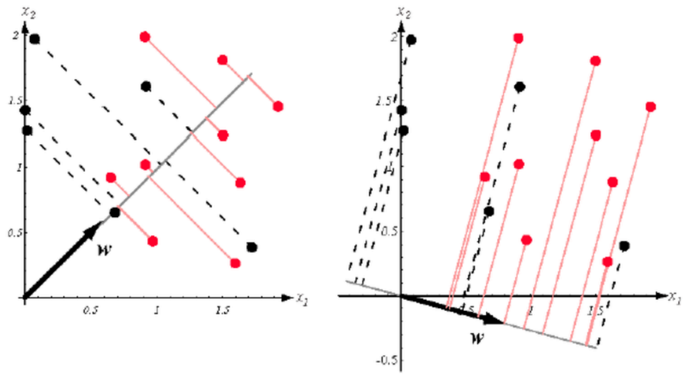

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[FMMS '16]

Results: State of the Art

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linear in k

Classic method:
through computing top-k eigenvectors explicitly

$\tilde{O}(k \cdot \mathrm{nnz} + \ldots)$

e.g. power method

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[AL '17, MMS '18]

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nearly-linear!

Our method:
through rational approximation of sign(x)

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PCP(PCR) solver runtime:

(unaccelerated) $$\tilde{O}(\boxed{\mathrm{nnz}} + d \cdot \mathrm{sr}(A) \cdot \kappa^2 \gamma^{-2})$$

(accelerated) $$\tilde{O}(\boxed{\mathrm{nnz}} + \sqrt{\mathrm{nnz} \cdot d \cdot \mathrm{sr}(A)} \cdot \kappa \gamma^{-1})$$

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Low-order for large n

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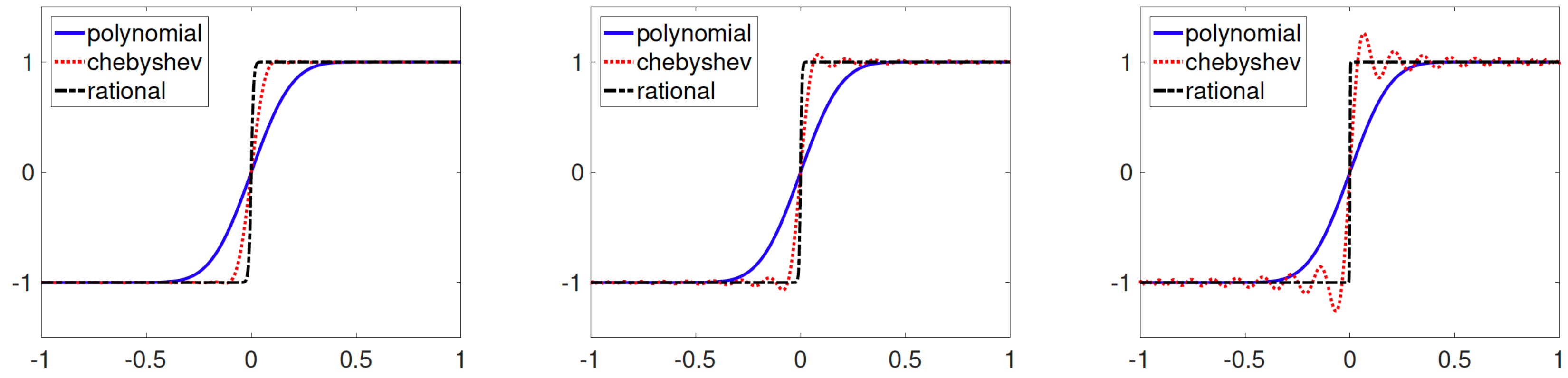
desired Zolotarev rational: $r_k^{\lambda\gamma}(x) = Cx \prod_{i \in [k]} \frac{x^2 + c_{2i}}{x^2 + c_{2i-1}} \approx \mathrm{sign}(x)$

[Zolotarev 1877]

Technique #1: Rational Approximation

"Stronger linear algebraic primitives solve problem in fewer steps"

- Same degree in comparison with standard polynomials [FMMS16], [AL17]



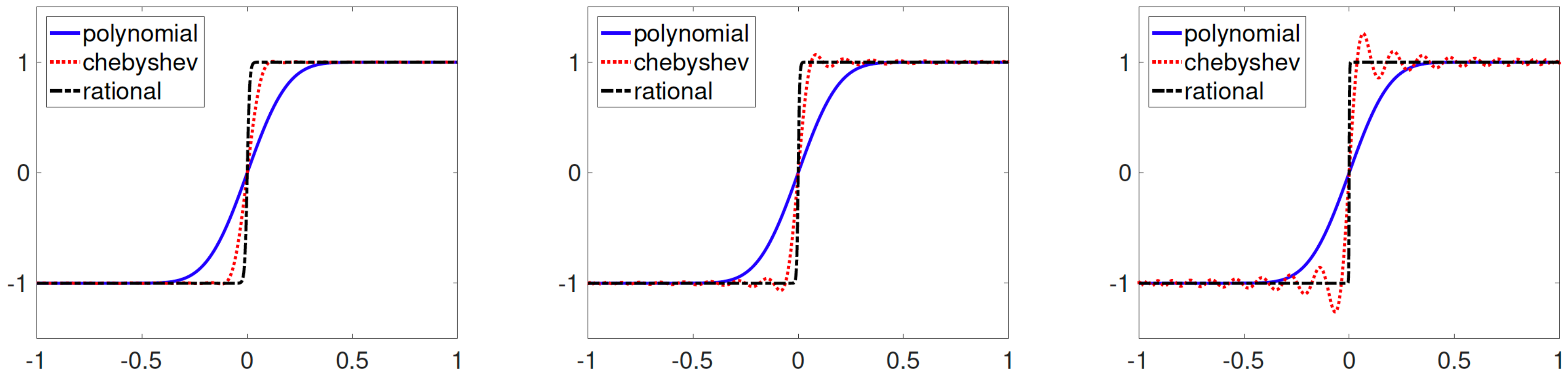
(a) γ=0.1 (b) γ=0.05 (c) γ=0.01

Figure 1: same degree = 21, different γ

Technique #1: Rational Approximation

"Stronger linear algebraic primitives solve problem in fewer steps"

- Compared with standard polynomials [FMMS16], [AL17]



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Better Approximation Quality under same degree

Technique #2: Asymmetric SVRG solver

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non-PSD

Technique #2: Asymmetric SVRG solver

"Variance reduction for non-convex problems"

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BUT
smaller variance!

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Our method for solving asymmetric

SVRG (variance-reduced stochastic gradient descent) $\widetilde{O}(\mathrm{nnz} + d \cdot \mathrm{sr}(A) \cdot 1/\mu^2)$

Savings:
Dimensional-related factor in runtime compared with solving squared systems directly

Results: main ideas

PCP: project a given vector onto $\mathcal{V}^\star$,
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1. Rational function approx.
reduce to solve $\tilde{O}(1)$
linear systems in form
$((A^\top A - \lambda I)^2 + cI)x = v$

2. Asymmetric SVRG applied
to nonconvex problem
Runtime: $\tilde{O}(\mathrm{nnz} + d \cdot \mathrm{sr}(A)\lambda^{-2}\gamma^{-2})$

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Overall complexity
(unaccelerated):

$$\tilde{O}(\mathrm{nnz} + d \cdot \mathrm{sr}(A) \cdot \frac{\kappa^2}{\gamma^2})$$

Takeaways:

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  - **III.** for nearly-linear time PCP and PCR solver

Thank you!

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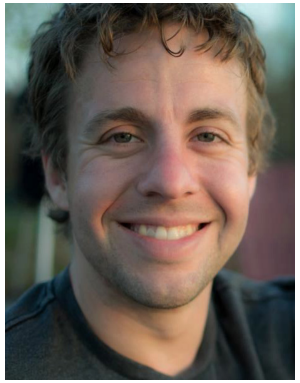
Aaron Sidford



Questions?

- Welcome to our poster @ 10:45am – 11:45am, Wednesday (12/11)
  @ East Exhibition Hall B + C #162
- arxiv: 1910.06517
- Email: yujiajin@stanford.edu